SUPPLEMENTARY MATERIAL

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I. Optimality of $\mathcal{P}_{\text{CIR},\Delta,s}$

Similar to $\mathcal{S}_\Delta$, we define the set of supports that satisfy the non-circular Δ-separation condition.

$$\mathcal{S}_{\text{non-cir},\Delta} = \{\mathbb{S} \subseteq \{1, \cdots, N\} | b - a > \Delta, \forall a < b \in \mathbb{S}\}$$

For a generic vector $\mathbf{x} \in \mathbb{C}^K$, the projection $\mathcal{P}_{\text{non-cir},\Delta,s}$ has been shown to be optimal in the following sense [11]

$$\mathcal{P}_{\text{non-cir},\Delta,s}(\mathbf{x}) \in \arg\min_{\mathbf{z}\in\mathbb{C}^K} \{\|\mathbf{z} - \mathbf{x}\|_2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_{\text{non-cir},\Delta}, |\text{supp}(\mathbf{z})| \leqslant s\}$$

We prove the optimality of $\mathcal{P}_{\text{cir},\Delta,s}$ by enumerating all possible cases and using the above optimality of $\mathcal{P}_{\text{non-cir},\Delta,s}$.

Let $\hat{\mathbf{x}}$ be any optimal approximation of $\mathbf{x} \in \mathbb{C}^N$ that

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{z}\in\mathbb{C}^N} \{\|\mathbf{z} - \mathbf{x}\|_2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_\Delta, |\text{supp}(\mathbf{z})| \leqslant s\}$$

Let $k = \max\{\text{supp}(\hat{\mathbf{x}})\}$ which denotes the largest index of the support of $\hat{\mathbf{x}}$. We consider two complementary cases:

- **Case 1:** If $1 \leqslant k \leqslant N - \Delta$, then we know $[\hat{\mathbf{x}}]_n = 0, k+1 \leqslant n \leqslant N$. By definition, we obviously have

  $$\hat{\mathbf{x}}_{1:N-\Delta} \in \arg\min_{\mathbf{z}\in\mathbb{C}^{N-\Delta}} \{\|\mathbf{z} - \mathbf{x}_{1:N-\Delta}\|_2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_{\text{non-cir},\Delta}, |\text{supp}(\mathbf{z})| \leqslant s\}$$

  Otherwise we can construct a strictly better approximation than $\hat{\mathbf{x}}$ using $\mathcal{P}_{\text{non-cir},\Delta,s}(\mathbf{x}_{1:N-\Delta})$ padded with zeros.
- **Case 2:** If $N - \Delta + 1 \leqslant k \leqslant N$, $\hat{\mathbf{x}}$ must be of the following structure

  $$\hat{\mathbf{x}} = [\underbrace{0, \cdots, 0}_{\Delta+k-N}, \cdots, \underbrace{0, \cdots, 0}_{\Delta}, \hat{x}_k, \underbrace{0, \cdots, 0}_{N-k}]^T$$

  Then we circularly shift $\hat{\mathbf{x}}$ to the left and obtain

  $$\hat{\mathbf{x}}^k = [\hat{x}_k, \underbrace{0, \cdots, 0}_{\Delta}, \cdots, \underbrace{0, \cdots, 0}_{\Delta}]^T$$

  Rotate $\mathbf{x}$ in the same way and get

  $$\mathbf{x}^k = [x_k, x_{k+1}, \cdots, x_N, x_1, \cdots, x_{k-1}]^T$$

  Because the projection $\mathcal{P}_{\text{cir},\Delta,s}$ is circularly rotation invariant, we must have that

  $$\hat{\mathbf{x}}^k \in \arg\min_{\mathbf{z}\in\mathbb{C}^N} \left\{\|\mathbf{z} - \mathbf{x}^k\|_2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_\Delta, |\text{supp}(\mathbf{z})| \leqslant s\right\}$$

  Then the argument in Case 1 implies that $\hat{\mathbf{x}}^k_{1:N-\Delta}$ satisfies

  $$\hat{\mathbf{x}}^k_{1:N-\Delta} \in \arg\min_{\mathbf{z}\in\mathbb{C}^{N-\Delta}} \left\{\|\mathbf{z} - \mathbf{x}^k_{1:N-\Delta}\|_2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_{\text{non-cir},\Delta}, |\text{supp}(\mathbf{z})| \leqslant s\right\}$$

Let $\gamma = \min_{\mathbf{z}\in\mathbb{C}^N} \left\{\|\mathbf{z} - \mathbf{x}\|_2^2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_\Delta, |\text{supp}(\mathbf{z})| \leqslant s\right\}$. Our earlier arguments imply that we must have

$$\gamma = \min_{\mathbf{z}\in\mathbb{C}^{N-\Delta}} \left\{\|\mathbf{z} - \mathbf{x}^k_{1:N-\Delta}\|_2^2 + \|\mathbf{x}^k_{N-\Delta+1:N}\|_2^2 | \text{supp}(\mathbf{z}) \in \mathcal{S}_{\text{non-cir},\Delta}, |\text{supp}(\mathbf{z})| \leqslant s\right\} \quad (1)$$

for some $N - \Delta + 1 \leqslant k \leqslant N$ or $k = 0$ ($\mathbf{x}^0 = \mathbf{x}$). Next, note that any element $\hat{\mathbf{v}}^k$ constructed in Table 1 is circularly Δ-separated. Then by (1), we must have $\hat{\mathbf{v}}^{k\#}$ will be one valid optimal approximation with circularly Δ-separated support.

II. Sep-ADMM

Table 3 Sep-ADMM

Input: $\mathbf{A}$, $\mathbf{y}$, s, ν, maxIter
Initialize $i = 0, \mathbf{u}^0 = \mathbf{0}$, $\mathbf{z}^0 = \mathbf{0}$
while $i <$ maxIter **do**
  $\mathbf{x}^{i+1} = (\mathbf{A}^H\mathbf{A} + \nu\mathbf{I})^{-1}(\mathbf{A}^H\mathbf{y} + \nu(\mathbf{z}^i - \mathbf{u}^i))$
  $\hat{\mathbf{z}}^{i+1}$:
$$\nu(\hat{\mathbf{z}}^{i+1} - \mathbf{x}^{i+1}) - \nu(\mathbf{u}^i) + \frac{1}{2}\partial\mathcal{Q}(\lambda\|\hat{\mathbf{z}}^{i+1}\|_0) = 0 \quad (2)$$
  $\mathbf{u}^{i+1} = \mathbf{u}^i + (\mathbf{x}^{i+1} - \hat{\mathbf{z}}^{i+1})$
  $\mathbf{z}^{i+1} = \mathcal{P}_{\text{cir},\Delta,s}(\hat{\mathbf{z}}^{i+1})$
end while
Output: The support estimate $\mathbb{S}^\# = \text{supp}(\mathbf{z}^{\text{maxIter}})$.

Using the sub-differential of $\mathcal{Q}(\cdot)$, the $\hat{\mathbf{z}}^{i+1}$ in (2) has the following closed-form solution

$$\hat{z}_n^{i+1} = \begin{cases} \frac{\nu(x_n^{i+1} + u_n^i)}{\nu - 1} - \frac{\sqrt{\lambda}}{\nu - 1}\frac{x_n^{i+1} + u_n^i}{|x_n^{i+1} + u_n^i|} & 0 \leqslant |x_n^{i+1} + u_n^i| \leqslant \sqrt{\lambda} \\ x_n^{i+1} + u_n^i & |x_n^{i+1} + u_n^i| > \sqrt{\lambda} \end{cases}$$

We set $\nu = 0.2$ in the simulations.