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Fractional-Order Control: Theory and Applications in Motion Control

This article carries on the discussion on fractional-order control started by the article D. Cafagna, "Fractional calculus: A mathematical tool from the past for present engineers," *IEEE Industrial Electronics Magazine*, vol. 1, no. 2, pp. 35–40, Summer 2007.

The concept of fractional-order control (FOC) means controlled systems and/or controllers are described by fractional-order differential equations. Expanding derivatives and integrals to fractional orders is by no means new and actually has a firm and long-standing theoretical foundation. Interest in this subject was evident almost as soon as the ideas of classical calculus were known. The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville, Holmgren, and Riemann, although Eular, Lagrange, and others made contribution even earlier [1]. Parallel to these theoretical beginnings was the development of applying fractional calculus to various problems [1].

As to fractional calculus' application in control engineering, FOC was introduced by Tustin for the position control of massive objects (see Figure 1) half a century ago in 1958, where actuator saturation requires sufficient phase margin around and below the critical point [2].

The characteristic equation of the above close-loop $1/s^{\beta}$ system with variable gain factor A is

$$1 + A(j\omega)^{\beta} = 0 \tag{1}$$

where $A = J_m/K_d$ in nominal case and $\beta = 2 - \alpha$. Equation (1) can be rewritten in the form

$$(j\omega)^{\beta} = -\frac{1}{A}. \tag{2}$$

The movement of $-1/A$ can be considered to be the locus of the critical point (see Figure 2) when the gain variation occurs. For integer order system, when $\beta = 2$, the system will be oscillatory due to its zero phase margin. Taking $\beta = 1$ leads to poor robustness against saturation since pure D controller will be used. By letting β be fractional between one and two, a better tradeoff between stability and robustness will be obtained. Namely, the fractional-order D^{α} controller is naturally introduced whose order α should be chosen properly between zero and one. Therefore, necessary phase margin can be easily kept to any desired amount in wide range of frequencies below and in the neighborhood of the critical point. This characteristic highlights the hopeful aspect of applying FOC to control engineering.

Some other pioneering works were also carried out around 1960s by Manabe [3]–[5]. However, the FOC concept was not widely incorporated into control engineering mainly due to the unfamiliar idea of taking fractional order, so few physical applications and limited computational power available at that time [6].

In last few decades, researchers pointed out that fractional-order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing complex dynamic features [1], [7]. Especially for the modeling and identification of flexible structures with increasing application of lighter materials, fractional- order differential equations could provide a natural solution since these structures are essentially distributed-parameter systems [8]. Obviously, the fractional-order models need fractional-order controllers for more effective control of the "real" systems. This necessity motivated renewed interest in various applications of FOC [10]–[13]. And with the rapid development of computer performances, realization of FOC systems also became possible and much easier than before.

Generally there are three main advantages for introducing fractional-order calculus to control engineering:

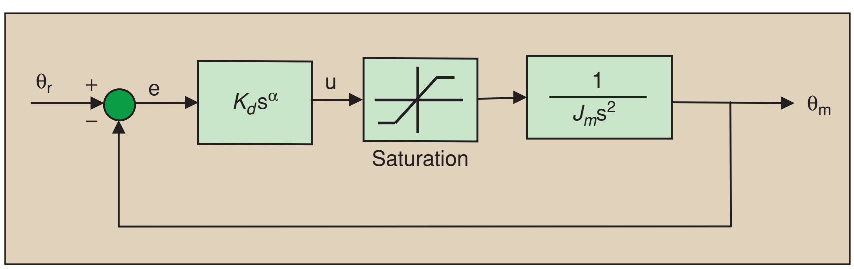

FIGURE 1 — The position control loop with fractional order D^{α} controller.

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1) adequate modeling of control plant's dynamic features
2) effective and clear-cut robust control design
3) reasonable realization by approximation.

In following sections, detailed descriptions of the above three advantages will be given.

Mathematical Aspects

Mathematic Definitions

One of the most frequently encountered definitions of fractional-order calculus is called the Grünwald-Letnikov definition:

$$ {}_aD_t^{\alpha} = \lim_{\substack{h\to 0\\ nh=t-\alpha}} h^{-\alpha}\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j} \times f(t-jh) \quad (3)$$

where $\binom{\alpha}{j}$ are the binomial coefficients. Obviously, introducing the fractional-order calculus leads to infinite dimension, while the integral calculus is finite dimension.

Another widely known definition is called the Riemann-Liouville definition with an integrodifferential expression. The definition for the fractional-order integral is

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1} \times f(\xi)d\xi \quad (4)$$

while the definition of fractional-order derivatives is

$$ {}_aD_t^{\alpha} f(t) = \frac{d^{\gamma}}{dt^{\gamma}}\left[{}_aD_t^{-(\gamma-\alpha)}\right] \quad (5)$$

where $\Gamma(x)$ is the Gamma function, a and t are limits, and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. γ is an integer that satisfies $\gamma - 1 < \alpha \le \gamma$. The Grünwald-Letnikov and Riemann-Liouville definitions are both a unification of integer order derivatives and integrals [7].

Laplace and Fourier Transforms

Fractional-order calculus is quite complicated in time domain, as shown in its two definitions. Fortunately one of the features most important to control engineers, its Laplace transform, is very straightforward [7]. The final expression of the Laplace transform of the fractional-order derivative is

$$ L\left\{{}_0D_t^{\alpha} f(t)\right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k_0 D_t^{\alpha-k-1} f(0) \quad (6)$$

where $n-1 < \alpha < n$ again. If all the initial conditions are zero, the Laplace transform of fractional-order derivative is simply

$$ L\left\{{}_0D_t^{\alpha} f(t)\right\} = s^{\alpha}F(s). \quad (7)$$

Therefore the Laplace transforms of fractional $\pm\alpha$ order calculus lead to the use of fractional-order Laplace operator $s^{\pm\alpha}$. The transfer functions of models and controllers, which are described by fractional-order differential equations, can be derived conveniently using fractional-order Laplace operator $s^{\pm\alpha}$.

Similarly, the Fourier transform of fractional-order derivative is

$$ F_e\left\{{}_0D_t^{\alpha} f(t)\right\} = (j\omega)^{\alpha}F(j\omega). \quad (8)$$

The frequency response of FOC system can be exactly obtained by substituting $s^{\pm\alpha}$ with $(j\omega)^{\pm\alpha}$ in its transfer function. This advantage implies frequency-domain analysis of FOC system is as convenient as integer order control (IOC) systems. The graphical tools for IOC in frequency domain are still available for FOC analysis and design.

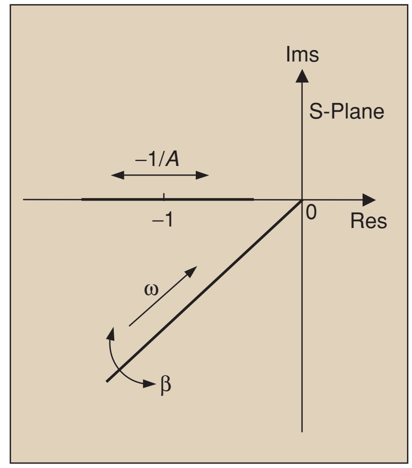

FIGURE 2 — Nyquist plots of the fractional order $1/s^{\beta}$ system.

Modeling and Identification

Recently, there has been a growing significant demand for better mathematic models to describe real objects. The fractional-order model can provide a new possibility to acquire more adequate modeling of dynamic processes. Fractional-order models have been applied to describe reheating furnaces [7], viscoelasticity [1], [7], [8], chemical processes [14], and chaos systems [15].

Actually, using a fractional-order model for describing distributed-parameter systems is quite natural since the Laplace transform of partial differential equations will inevitably introduce the fractional-order s operator. For a simple example, consider a torsional model as shown in Figure 3, which consists of a flexible shaft attached to a rigid disk [16]. The rigid body equation of the disk is given as

$$ I_1 s^2\theta_1 = T_1 + T_{12}. \quad (9)$$

Take a small element of length dx along the shaft axis and observe the cylindrical surface, as shown in

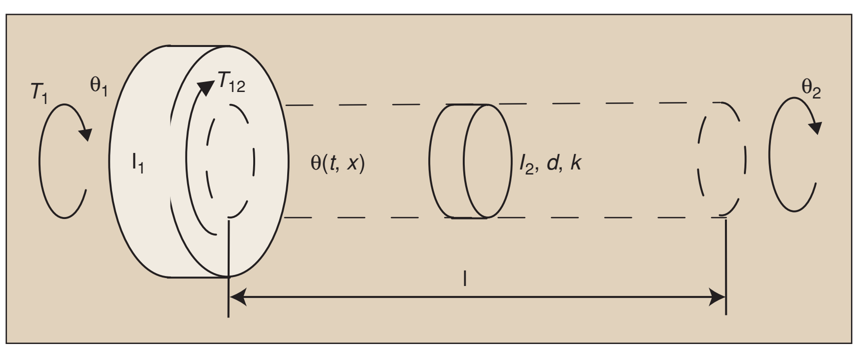

FIGURE 3 — The flexible shaft attached to a rigid disk.

Figure 4(a). This element will deform through a small angle $d\theta$.

Based on the theory of elasticity, the corresponding shear stress at the deformed point at radius r is

$$\tau = G\gamma = Gr\frac{\partial\theta(t,x)}{\partial x} \tag{10}$$

where G is shear modulus and γ is the shear strain [17]. As shown in Figure 4(b), since this shear stress acts tangentially, the overall torque at the shaft cross section is

$$\begin{aligned} T &= \int r \times (\tau \times 2\pi r dr) \\ &= G\frac{\partial\theta(t,x)}{\partial x}\int 2\pi r^3 dr \\ &= GJ\frac{\partial\theta(t,x)}{\partial x}. \end{aligned} \tag{11}$$

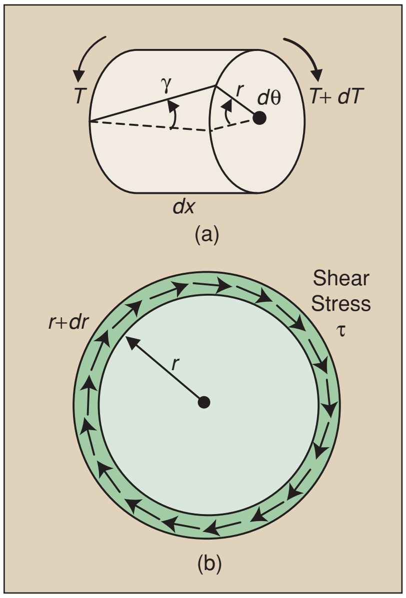

FIGURE 4 — Deformation of the torsional shaft: (a) small element of the torsional shaft and (b) shear stress in a small annular cross section.

Apply Newton's second law for rotatory motion of the small element dx shown in Figure 4(a), the equation of motion is

$$\begin{aligned} \rho J dx\frac{\partial^2\theta(t,x)}{\partial t^2} &= T + dT - T \\ &= \frac{\partial T(t,x)}{\partial x}dx. \end{aligned} \tag{12}$$

For a uniform shaft segment of length l with associated overall angular deformation θ, the torsional stiffness k is

$$k = \frac{T}{\theta} = GJ\frac{\partial\theta(t,x)}{\partial x}\cdot\frac{1}{\theta} = \frac{GJ}{l}. \tag{13}$$

Based on (11) and (12), the following equation can be obtained:

$$\frac{I_2}{l}\frac{\partial^2\theta(t,x)}{\partial t^2} - kl\frac{\partial^2\theta(t,x)}{\partial x^2} = 0. \tag{14}$$

For (14), the Laplace transform in t, a second-order differential equation, is

$$\frac{I_2}{l}s^2\theta(x) - kl\frac{d^2\theta(x)}{dx^2} = 0 \tag{15}$$

where $\theta(s,x)$ is abbreviated as $\theta(x)$ for simplicity. For the free end of the shaft, there is no deformation and the shear stress is zero. Therefore, the following boundary conditions can be obtained:

$$\theta(x)\Big|_{x=0} = \theta_1, \quad \frac{d\theta(x)}{dx}\Big|_{x=l} = 0. \tag{16}$$

Torque T_{12} in Figure 3 can be obtained:

$$\begin{aligned} T_{12}(s) &= kl\frac{d\theta(x)}{dx}\Big|_{x=0} \\ &= -\tanh(\mu ls)\theta_1 \end{aligned} \tag{17}$$

where $\mu^2 = I_2/kl$. Finally, substitute T_{12} in (9), the transfer function between T_1 and θ_1 can be achieved:

$$\frac{T_1}{\theta_1} = I_1 s^2 + kl\cdot\mu s\cdot\tanh(\mu ls). \tag{18}$$

However, in a conventional modeling method, the torsional system in Figure 3 is usually modeled as a rigid body system with inertia $I = I_1 + I_2$:

$$\frac{T_1}{\theta_1} = (I_1 + I_2)s^2. \tag{19}$$

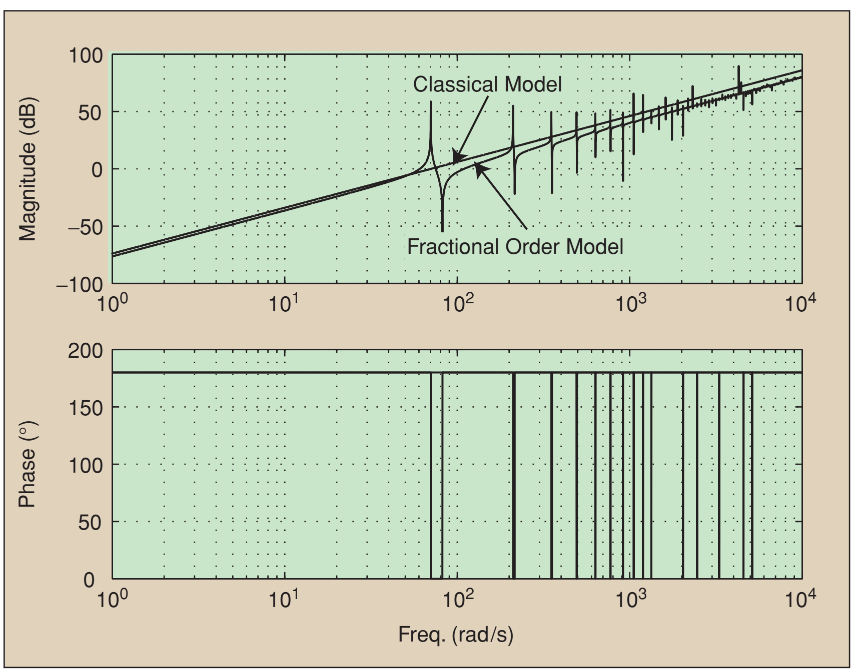

FIGURE 5— Bode plots of the torsional system's fractional-order model and conventional integer order model.

As shown in the Bode plots of Figure 5, the fractional-order transfer function model in (18) displays the mechanical resonance effect naturally. At low frequency range, the two models give similar frequency responses. At high frequency range, the fractional model can describe the distributed nature of the torsional system; while in conventional integer order model, this nature is totally ignored. Fractional-order modeling is a useful tool to give a more adequate description of system's "real" dynamic features.

From the above example, it can be seen that distributed-parameter systems are naturally described by a set of partial differential equations. However, these equations will lead to transfer functions that are quotients of transcendental functions.

Using a fractional-order transfer function model, a quotient of polynomials in

s^α, it is also possible to fit better a set of experimental data. For example, the frequency-domain identification of a flexible structure by a fractional-order model can take into account not only material damping, but also other varieties of physical phenomena such as viscoelasticity and anomalous relaxation. This fact indicts fractional-order models can be an appropriate and hopeful tool to model the dynamic features of flexible structure more accurately which is becoming more and more important due to lighter materials and faster motions [7], [8].

For fractional-order models like (20), frequency-domain identification methods to determine the coefficients $\alpha_k, \beta_k (k = 0, 1, 2, \ldots)$ and $a_k, b_k (k = 0, 1, 2, \ldots)$ are as routine as conventional integer order models.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + \cdots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \cdots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}. \quad (20)$$

Various identification methods for determination of the coefficients were developed [7]–[9], based on minimization of the difference between the measured frequency response $F(\omega)$ and the frequency response of the model $G(j\omega)$. For example, the quadratic criterion for the optimization can be in following form:

$$Q = \sum_{m=0}^{M} W^2(\omega_m) |F(\omega_m) - G(j\omega_m)|^2 \quad (21)$$

where $W(\omega_m)$ is a weighting function and M is the number of measured values of frequencies $\omega = (\omega_0, \omega_1, \ldots, \omega_M)$.

Compared to the general fractional-order model as in (20), a special model can be introduced, in which only integer orders of the fractional-order operator s^α are used:

$$G(s) = \frac{\sum_{i=0}^{m} a_i (s^\alpha)^i}{(s^\alpha)^n + \sum_{j=0}^{n-1} b_j (s^\alpha)^j}, \quad n \geq m. \quad (22)$$

It is interesting to note that the selection of α can actually be seen as selecting the phenomena that can be modeled. For example, when modeling a flexible structure, using $\alpha = 2$ can not model damping. In $\alpha = 1$ case, the damping can be modeled. By further taking $\alpha = 0.5$, other phenomena such as viscoelasticity and anomalous relaxation will be described. The other advantage of this model is that existing optimization methods can still be used since only integer order s^α is introduced.

Realization Methods

Though it is not difficult to understand the theoretical advantages of FOC, especially in frequency domain, realization issue kept being somewhat problematic and perhaps was one of the most doubtful points for the application of FOC. Fractional-order systems have an infinite dimension; while the conventional integer order systems are finite dimension. To realize fractional-order controllers perfectly, all the past inputs should be memorized. It is impossible in real applications. Proper approximation by finite differential or difference equation must be introduced.

A frequency-band, fractional-order controller can be realized by broken line approximation in frequency domain. But further discretization is required for this method [18]–[20]. As to direct discretization, various methods have been proposed such as sampling time scaling [21], short memory principle [7], Tustin Taylor expansion [22], Lagrange function interpolation method [10], while all the approximation methods need truncation of the expansion series. A detailed comparison of the above direct discretization methods can be found in [23].

Frequency-Band Approximation

Since a fractional-order system's frequency responses can be exactly known, approximating fractional-order controllers by frequency-domain approaches is natural. At the same time, it is neither practicable nor desirable to try to make the order be fractional in whole frequency range. The frequency-band, fractional-order controllers are required and practical in most control applications. The broken-line approximation method can be introduced to realize frequency-band, fractional-order I^α controller. Let

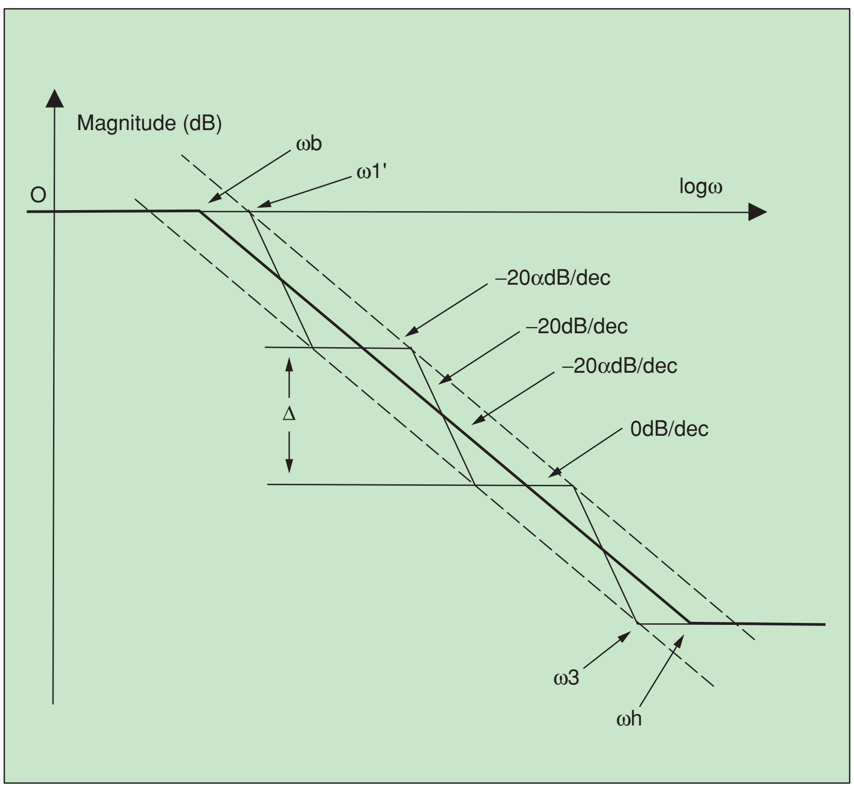

FIGURE 6— An example of broken-line approximation ($N = 3$).

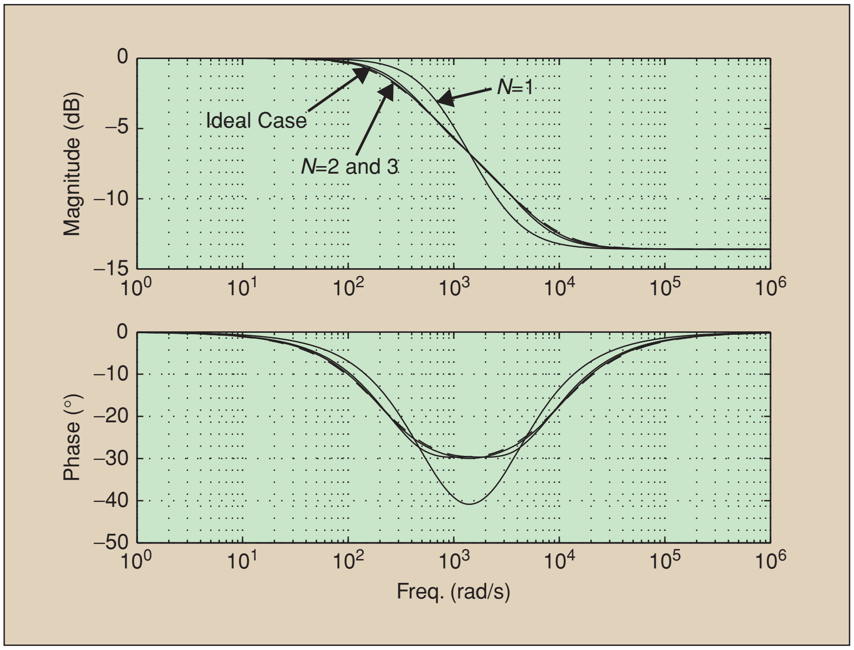

FIGURE 7— Bode plots of ideal case, the first-, second-, and third-order approximations.

$$\left(\frac{\frac{s}{\omega_h}+1}{\frac{s}{\omega_b}+1}\right)^{\alpha} \approx \prod_{i=0}^{N-1} \frac{\frac{s}{\omega_i'}+1}{\frac{s}{\omega_i}+1}. \quad (23)$$

Based on Figure 6, ω_i and ω_i' can be calculated as

$$\omega_i = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}-\frac{\alpha}{2}}{N}} \omega_b,$$

$$\omega_i' = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}+\frac{\alpha}{2}}{N}} \omega_b. \quad (24)$$

Figure 7 shows the Bode plots of ideal frequency-band case ($\alpha = 0.4, \omega_b = 200$ Hz, $\omega_h = 10,000$ Hz) and its first-, second-, and third-order approximations by broken-line approximation method. Even taking $N = 2$ can give a satisfactory accuracy in frequency domain.

Direct Discretization

The most commonly used discretization method of a fractional-order controller is called the short memory principle method. This discretization method is based on the observation that for the Grünwald-Letnikov definition, the values of the binomial coefficients near "starting point" $t = 0$ are small enough to be neglected or "forgotten" for large t. Therefore the principle takes into account the behavior of $x(t)$ only in "recent past," i.e., in the interval $[t - L, t]$, where L is the length of "memory"

$$_0D_t^{\alpha} x(t) \approx_{t-L} D_t^{\alpha} x(t), \ (t > L). \quad (25)$$

Based on approximation of the time increment h through the sampling time T, the discrete equivalent of the fractional-order α derivative is given by

$$Z\{D^{\alpha}[x(t)]\} \approx \left(\frac{1}{T^{\alpha}} \sum_{j=0}^{m} c_j z^{-j}\right) X(z) \quad (26)$$

where $m = [L/T]$ and the coefficients c_j are

$$c_0 = 1,$$
$$c_j = (-1)^j \binom{\alpha}{j}$$
$$= \frac{j-\alpha-1}{j} \cdot c_{j-1}, \quad j \geq 1. \quad (27)$$

It must be pointed out that the necessary memory length, namely how good the approximation is needed, should be decided by the demand of specific control problem [23]. Larger memory gives better performance but also leads to a longer computation time. However, this tradeoff is not restricted in FOC, actually a common problem in digital control.

An Example in Motion Control

Here a factional order PID^k controller is applied to torsional system's backlash vibration suppression control [24]. In PID^k controller D's order can

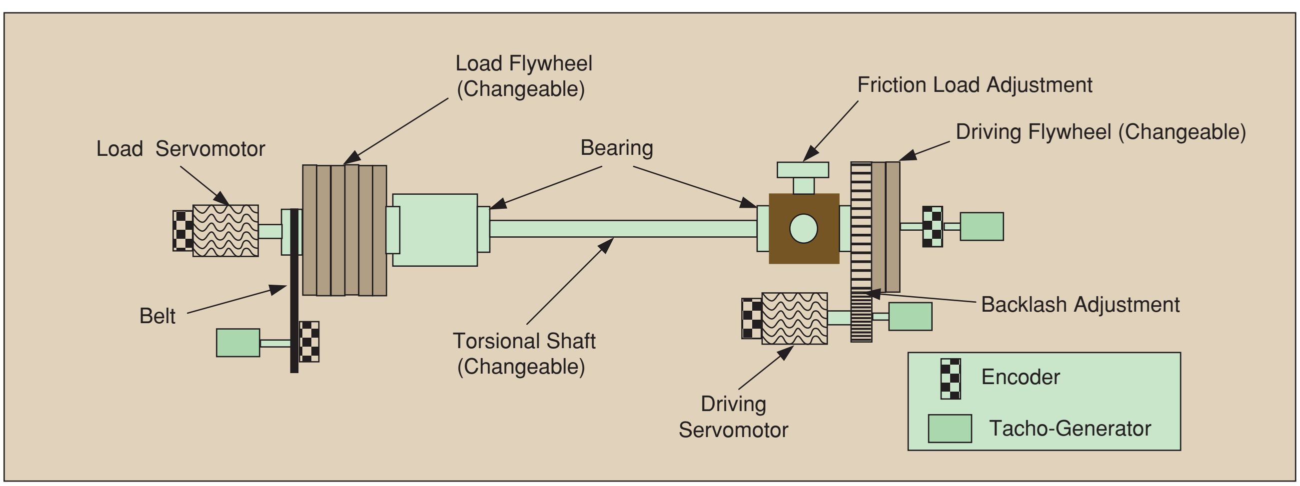

FIGURE 8— Experimental setup of torsional system.

be any real number, not necessarily be an integer. The experimental setup of torsional system is shown in Figure 8 where gear backlash exists. Tuning fractional-order k can adjust control systems frequency response directly; therefore a straightforward design can be achieved for robust control against backlash nonlinearity.

The simplest model of the torsional system with gear backlash is the three-inertia model shown in Figure 9, where J_m, J_g, and J_l are driving motor, gear (driving flywheel) and load's inertias, K_s shaft elastic coefficient, ω_m and ω_l motor and load rotation speeds, T_m input torque, and T_l disturbance torque. An interesting and more thorough analysis of backlash nonlinearity based on the describing function method can be found in [25], in which the fractional-order dynamics of the backlash is illustrated.

Since the gear elastic coefficient K_g is much larger than the shaft elastic coefficient K_s ($K_g >> K_s$), for speed control design the two-inertia model is commonly used in which driving motor inertia J_m and gear inertia J_g are simplified to a single inertia $J_{mg}(= J_m + J_g)$ (see Figure 10).

The PID controller is designed based on the simplified two-inertia model. Simulation results with the simplified two-inertia model show this integer order PID control system has a superior performance for suppressing torsion vibration (see Figure 11).

For three-inertia plant $P_{3m}(s)$, the close-loop transfer function of integer order PID control system from ω_r to ω_m is

$$G_{\text{close}}(s) = \frac{C_I(s)P_{3m}(s)}{1 + C_I(s)P_{3m}(s) + C_{PD}(s)P_{3m}(s)} \tag{28}$$

where $C_I(s)$ is I controller and $C_{PD}(s)$ is the parallel of P and D controllers in minor loop; therefore $G_{\text{close}}(s)$ is stable if and only if $G_l(s) = C_I(s)P_{3m}(s) + C_{PD}(s)P_{3m}(s)$ has positive gain margin and phase margin. But as shown in Figure 12 the gain margin of $G_l(s)$ is negative. With

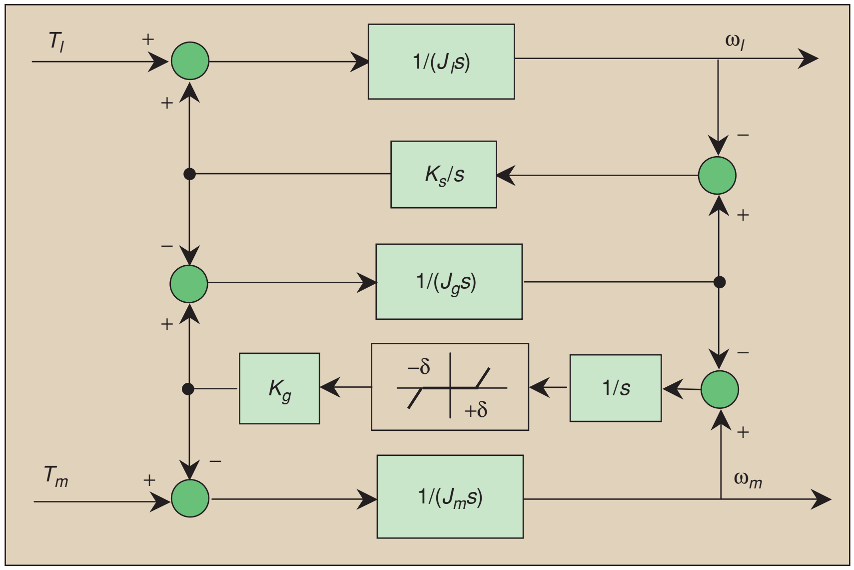

FIGURE 9— Block diagram of the three-inertia model.

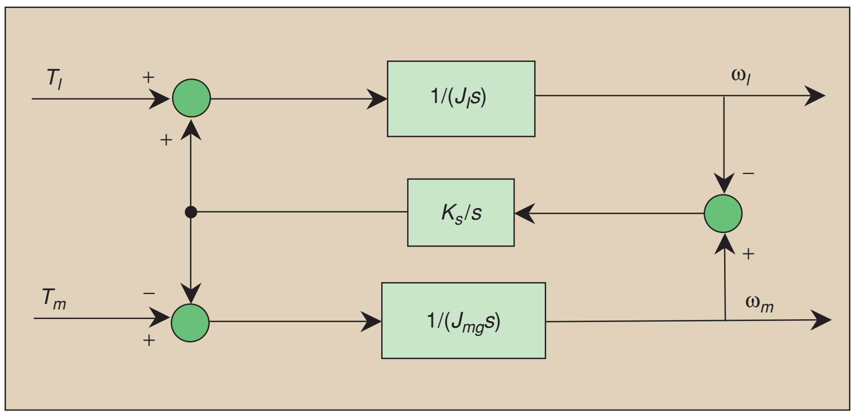

FIGURE 10— Block diagram of the two-inertia model.

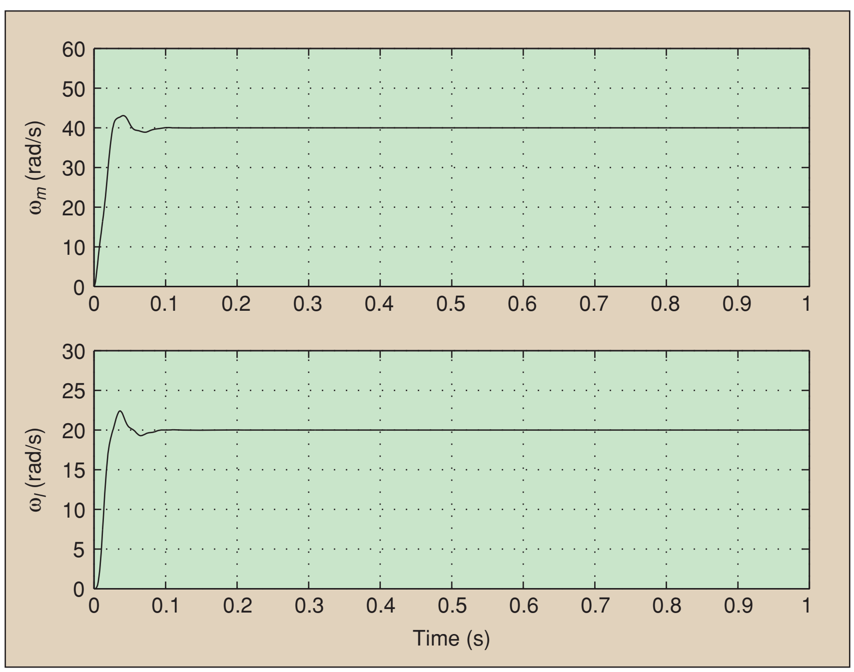

FIGURE 11— Time responses of the integer order PID two-inertia system by simulation.

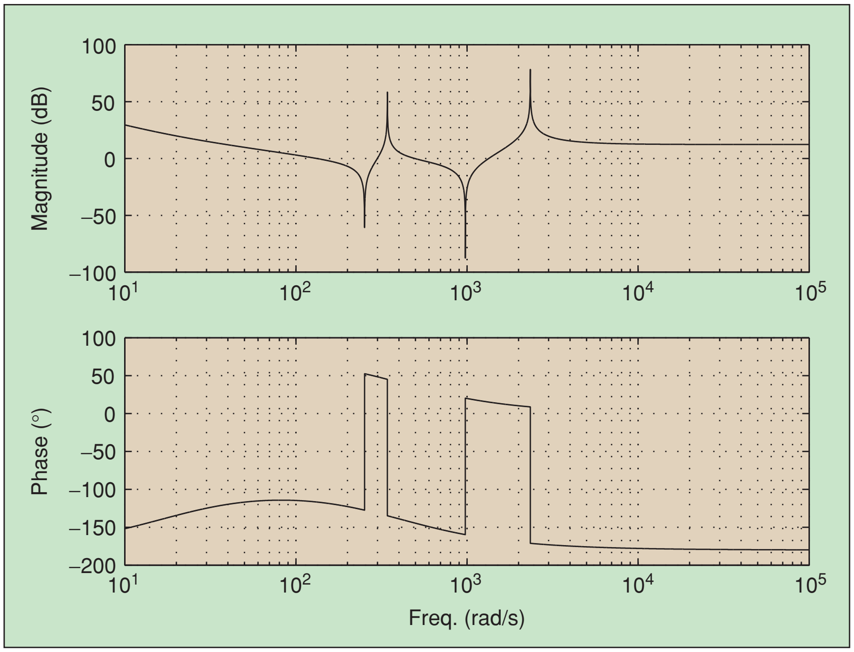

FIGURE 12— Bode plot of $G_l(s)$ in PID control.

the existence of gear backlash the designed integer order PID control system will easily be unstable and lead to backlash vibration.

To be robust against backlash non-linearity, several methods have been proposed, but their design processes are very complicated. As an example, for PID control introducing a low-pass filter $K_d s/(T_d s+1)$ and redesigning the whole control system with three-inertia model can be a solution [26]. Due to the necessity of solving high order equations, the design is not easy to carry out. On the contrary, fractional-order PID^k controller can achieve a straightforward design of robust control system against gear backlash non-linearity. By changing the D^k controller's fractional-order k the frequency response of $G_l(s)$

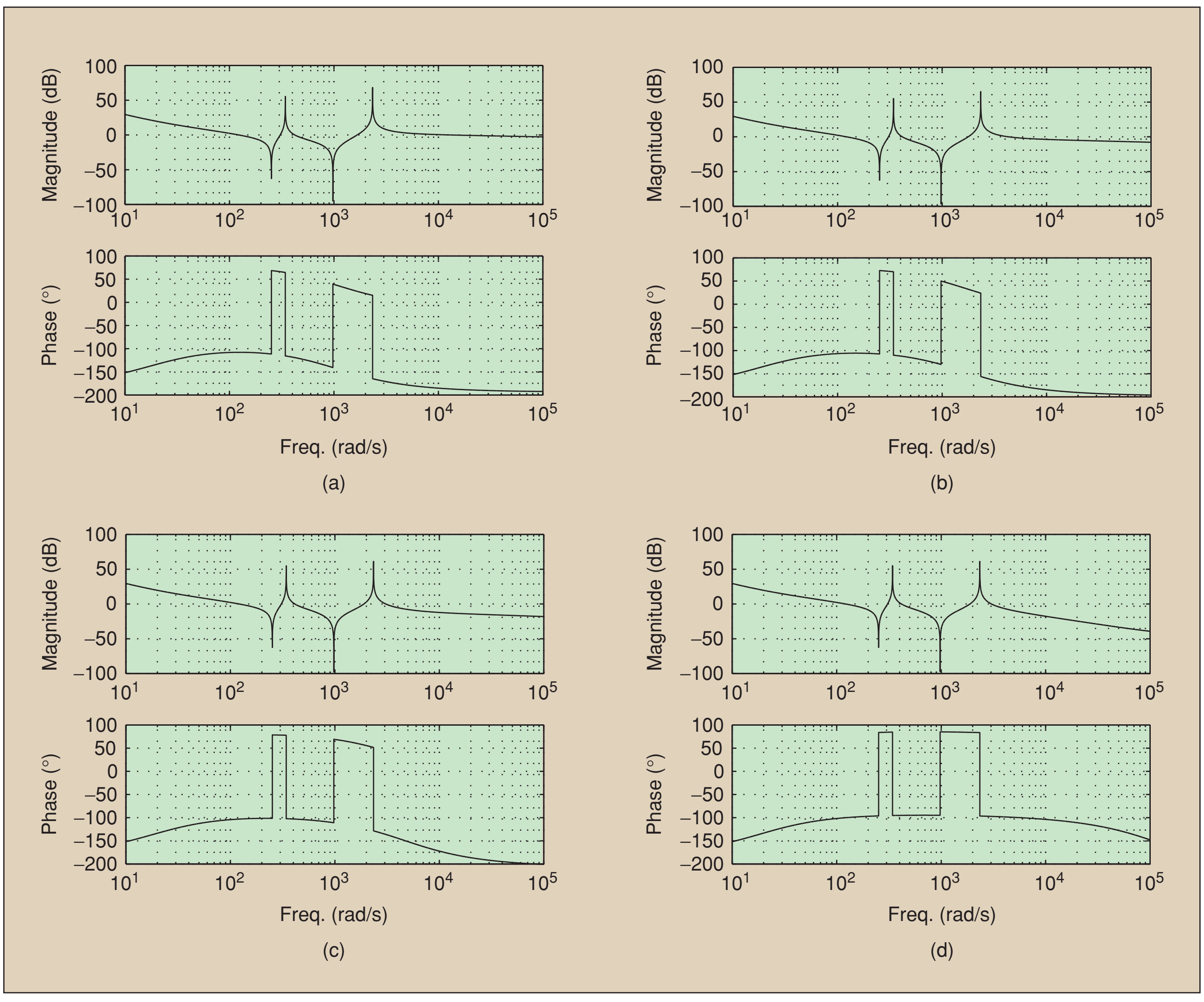

FIGURE 13— Bode plots of $G_l(s)$ in PID^k control: (a) $k=0.85$, (b) $k=0.8$, (c) $k=0.7$, and (d) $k=0.5$.

can be directly adjusted (see Figure 13). As shown in Figure 14, letting k be fractional order can improve PID^k control system's gain margin continuously. When $k < 0.84$ the PID^k control system will be stable; therefore with proper selected fractional-order k the backlash vibration can be suppressed.

At the same time, for better backlash vibration suppression performance higher D^k controller's order is more preferable. As shown in open-loop gain plots of 0.85, 0.8, 0.7 and 0.5 order PID^k control systems (see Figure 15), higher the D controller's order is taken lower the gain near gear backlash vibration mode is. Based on the tradeoff between robustness and vibration suppression performance, fractional-order 0.7 is chosen as D^k controller's best order. Here the short memory principle method is used to realize the discrete D^k controller.

As to the experimental results, first, integer order PID speed control experiment is carried out. As shown in Figure 16 the PID control system can achieve satisfactory response when the backlash angle is adjusted to zero degree ($\delta = 0$), while severe vibration occurs due to the existence of backlash nonlinearity (see $\delta = 0.6$ case). This experimental result is consistent with the above analysis.

Figure 17 shows the experimental results of fractional-order PID^k control with 0.7 and 0.5 order D^k controllers. Severe backlash vibration in the integer order PID control case is effectively suppressed. The control system's stability and robustness against gear backlash nonlinearity can be greatly improved by the FOC approach. PID$^{0.7}$ control system has a good robustness against backlash nonlinearity, while the error in the experimental response curves is for encoders' coarse quantization. The intermittent tiny vibrations in lower order 0.5 case are due to its relatively high gain near gear backlash vibration mode in open-loop frequency response.

It is interesting to find the vibration suppression performance of fractional-order PID^k control system shows somewhat "interpolation" characteristic. As shown in Figure 18, PID^1 control has the most severe backlash vibration, while PID$^{0.85}$ is on the verge of instability. PID$^{0.95}$ and PID$^{0.9}$ have intermediate time responses. This experimental result is natural since these orders are continuous. The "interpolation" characteristic is one of the main points to understand the superiority of FOC as providing more flexibility for designing robust control systems.

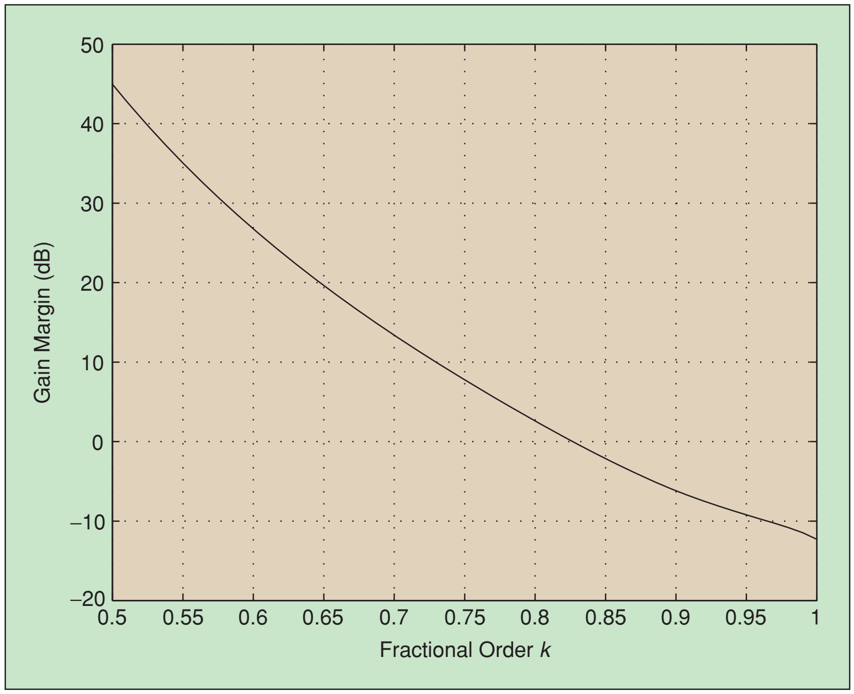

FIGURE 14— Gain margin versus fractional order k.

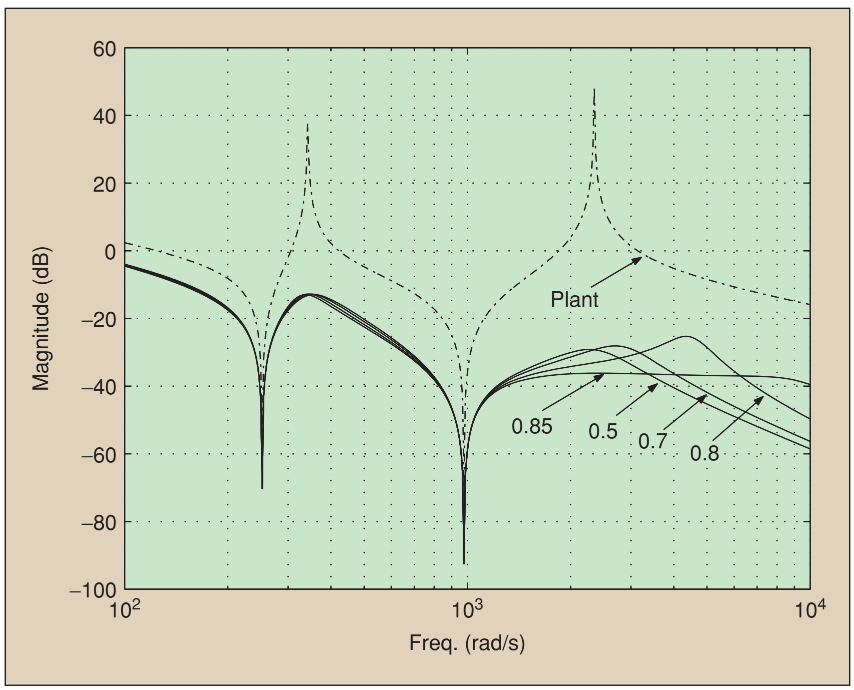

FIGURE 15— Gain plots of the PID^k control systems and three-inertia plant.

Conclusions

FOC opened a new dimension for control theory. The highly developed control theory based on integer order differential equations shows quite different characteristics when it is

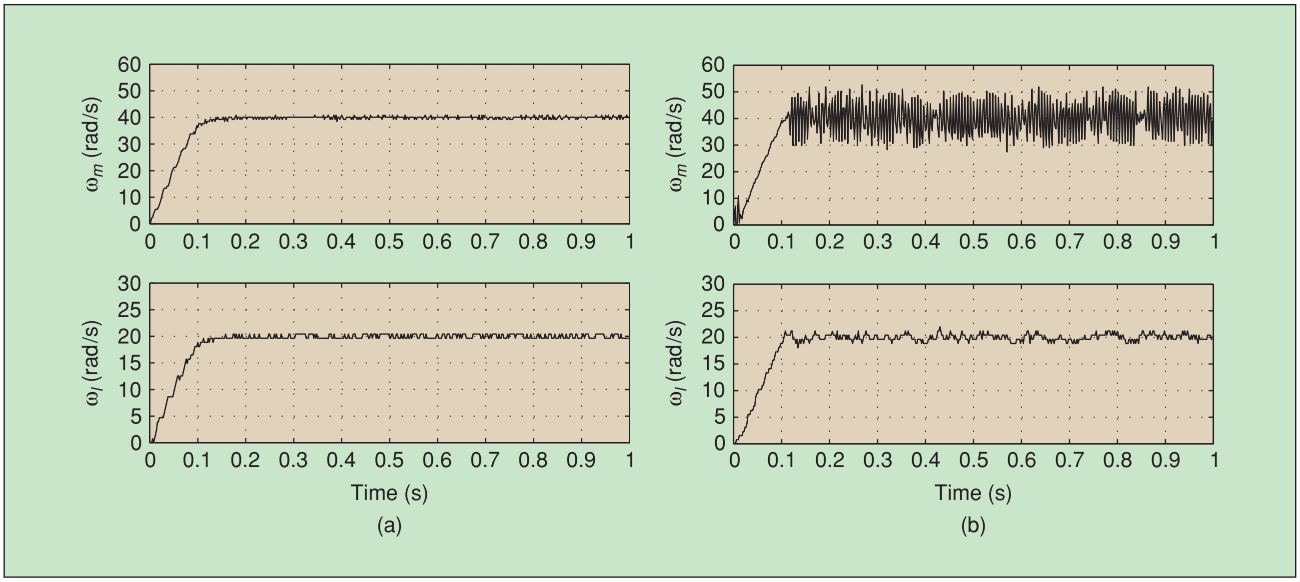

FIGURE 16— Time responses of the integer order PID control: (a) $\delta = 0^\circ$ and (b) $\delta = 0.6^\circ$.

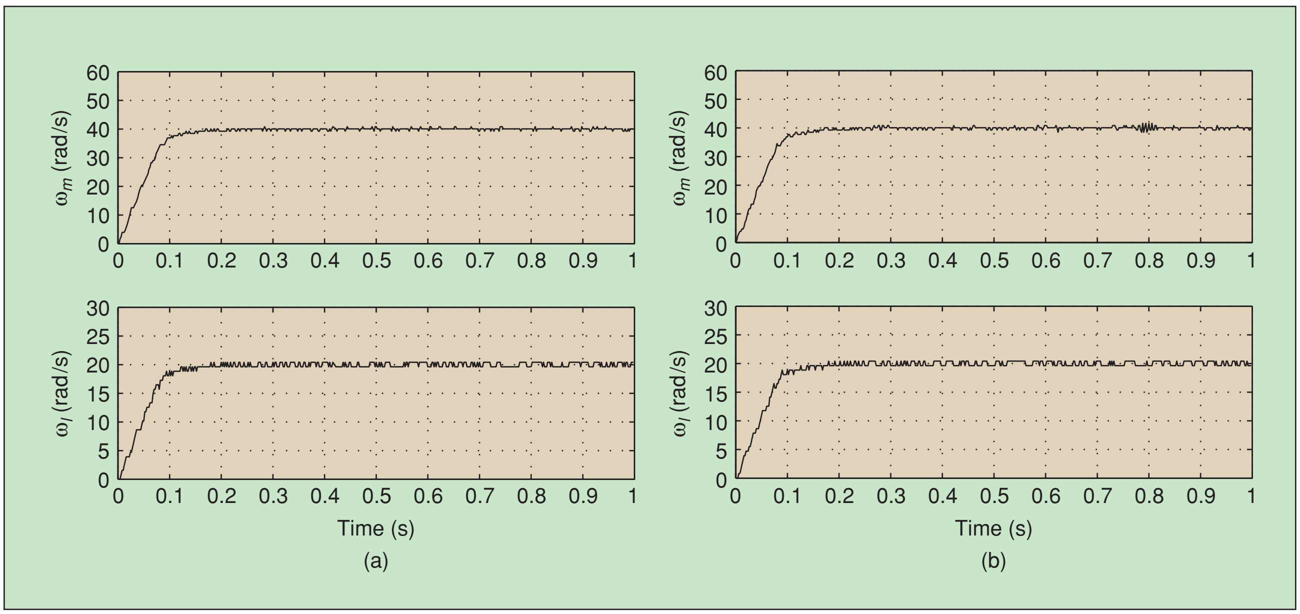

FIGURE 17— Time responses of PID^k control: (a) $k = 0.7$ and (b) $k = 0.5$.

expanded into a fractional-order field. At the same time, FOC is actually a nice generalization of IOC theory. This generalization gives huge space for researchers to see conventional IOC theory in a fresh light and find new and interesting things.

From a practice viewpoint, the ideal fractional-order controllers can only be realized by proper approximation with finite differential or difference equations. Namely, "design by FOC and realize by IOC" are inevitable. The practical advantages for FOC is to provide more flexibility and insight in control design and thus give a clear-cut approach for designing robust control system. The authors do believe some well-designed IOC system might in fact be a unconscious approximation of FOC system.

And the dynamic features of "real" systems can be described more adequately by fractional-order models. Especially for light materials and flexible structures, not only damping, but also other variety of physical phenomena such as viscoelasticity and anomalous relaxation should be taken into account. This demand naturally needs fractional-order models and hence fractional-order controllers, which are hopeful tools for modeling and controlling complex dynamic features.

Finally, the authors would like to end this introduction of FOC with the following expressive quotation:

"… all systems need a fractional time derivative in the equations that describe them … systems have memory of all earlier events. It is necessary

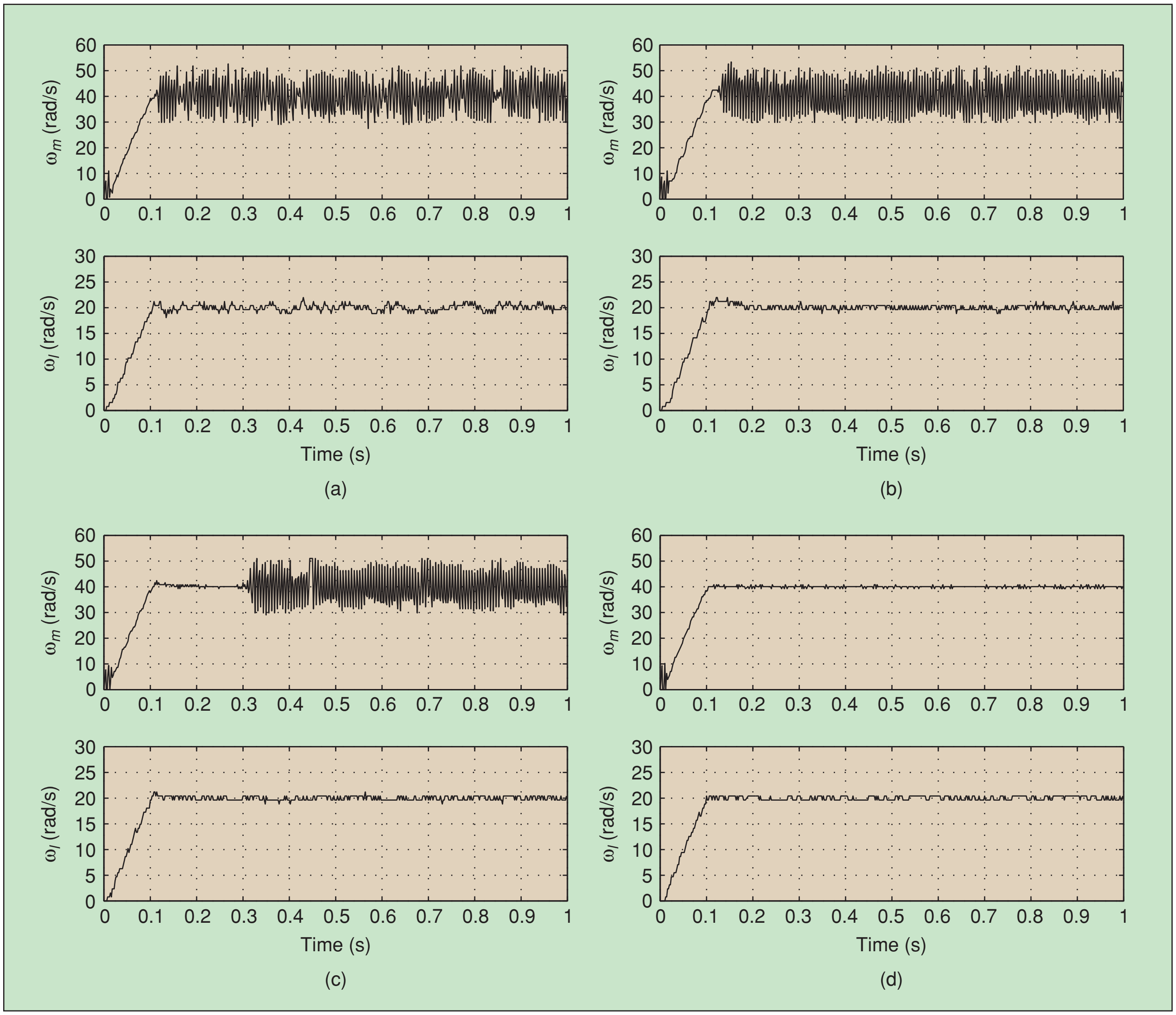

FIGURE 18— Continuity of PID^k control's vibration suppression performance: (a) $k = 1$, (b) $k = 0.95$, (c) $k = 0.9$, and (d) $k = 0.85$.

to include this record of earlier events to predict the future …

The conclusion is obvious and unavoidable, Dead matter has memory. Expressed differently, we may say that nature works with fractional time derivatives."—S. Westerlund, "Dead matter has memory!," *Physics Scripta,* vol. 43, pp. 174–179, 1991.

With fractional-order calculus and control, we may be able to extend a lot of new things.

Biographies

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