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Nonlinear Dynamics of Duffing System With Fractional Order Damping

In this paper, nonlinear dynamics of Duffing system with fractional order damping is investigated. The fourth-order Runge–Kutta method and tenth-order CFE-Euler method are introduced to simulate the fractional order Duffing equations. The effect of taking fractional order on system dynamics is investigated using phase diagram, bifurcation diagram and Poincaré map. The bifurcation diagram is introduced to exam the effect of excitation amplitude, frequency, and damping coefficient on the Duffing system with fractional order damping. The analysis results show that the fractional order damped Duffing system exhibits periodic motion, chaos, periodic motion, chaos, and periodic motion in turn when the fractional order varies from 0.1 to 2.0. The period doubling bifurcation route to chaos and inverse period doubling bifurcation out of chaos are clearly observed in the bifurcation diagrams with various excitation amplitude, frequency, and damping coefficient. [DOI: 10.1115/1.4002092]

1 Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number or complex number orders of the differential and integration operators. Since fractional calculus has a profound impact on many engineering and scientific areas such as automatic control, signal and image processing, bioengineering, electrochemistry, mechanics, viscoelasticity, and rheology, the applications of fractional calculus in engineering and physics have attracted lots of interest internationally [1–3]. Especially, the fractional calculus based modeling of complicated dynamics is becoming a recent focus of research. The dynamics of fractional order Chua, Lorenz, Rossler, Chen, Jerk, and Duffing were mainly investigated [4–10]. Obviously, the chaotic attractors for the fractional order systems should also have various fractional orders. The existing researches include the discussions on the effect of fractional order damping on the chaotic dynamics of Duffing equation [11], the bifurcation and chaotic dynamics of the fractional order cellular neural networks [12,13], the fractionally damped Van der Pol equation with periodical excitation [14,15], etc. It has been shown that the chaotic motion exists when the order of fractional damping is less than 1.

In recent years, the dynamics and vibration analysis of fractional order damped systems are of great interest to researchers [16–21]. The fractional order operator's characteristics of having an unlimited memory leads to concise and more adequate descriptions of complicated dynamics [22–24]. The Duffing equation, which is being used in many physical, mechanical, and even biological engineering problems, has been modified to study the dynamics of fractional order systems [5,6,11]. However, the existing researches mainly focus on the effect of the fractional order damping. The effect of other parameters including the damping coefficient, amplitude, and frequency of the external exciting force has not been investigated. Because these parameters also play an important role in the dynamic characteristics of the fractional order system, it is necessary to study the impact of the above parameters on the fractional dynamics.

Bearing these ideas in mind, this paper discusses the nonlinear analysis of fractionally damped Duffing system with the variation in not only the fractional order but also the damping coefficient, amplitude, and frequency of the external exciting force. An appropriate approximation of fractional order operator need to be introduced for the analysis of fractional order system's dynamics due to its unlimited dimension. A linear approximation of fractional order transfer function in frequency domain can be adopted to study the chaotic characteristics [4,7]. However, it is found that the approximated model obtained by frequency domain methods exhibits chaos whereas the original system is not actually chaotic [25]. In this paper, the direct approximation using Euler rule and continued fraction expansion (CFE) is introduced for the numerical simulation of fractional Duffing system.

2 Fractional Calculus and Discretization Schemes

The two definitions for fractional differentiation and integration are the Grünwald–Letnikov (GL) definition and Riemann–Liouville (RL) definition [26]. The GL definition is well-known for the discretization of fractional order operators. The GL definition is given by

$$ {}_aD_t^{\alpha} = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha}\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j}f(t-jh) \tag{1}$$

where the binomial coefficients are

$$\binom{\alpha}{0}=1,\quad \binom{\alpha}{j}=\frac{\alpha(\alpha-1),\ldots,(\alpha-j+1)}{j!}\quad \text{for } j\geq 1 \tag{2}$$

While the RL definition is given with an integrodifferential expression. The definition for fractional order integral is

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$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1}f(\xi)d(\xi) \tag{3} $$

while the definition of fractional order derivatives is

$$ {}_aD_t^{\alpha}f(t) = \frac{d^\gamma}{dt^\gamma}[{}_aD_t^{-(\gamma-\alpha)}f(t)] \tag{4} $$

where

$$ \Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy \tag{5} $$

is the Gamma function, a and t are the limits, and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. γ is an integer that satisfies $\gamma - 1 < \alpha \le \gamma$.

Fractional order calculus is quite complicated in time domain, as shown in the above two definitions. However, its Laplace transform is very straightforward. The Laplace transform of the fractional order integral of $f(t)$ is given by

$$ L\{{}_0D_t^{-\alpha}f(t)\} = s^{-\alpha}F(s) \tag{6} $$

where $F(s)$ is the Laplace transform of $f(t)$. While the Laplace transform of the fractional order derivative is

$$ L\{{}_0D_t^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k D_t^{\alpha-k-1}f(0) \tag{7} $$

where $n-1 < \alpha < n$ again. If all the initial conditions are zeros, the Laplace transform of fractional order derivative is simply

$$ L\{{}_0D_t^{\alpha}f(t)\} = s^{\alpha}F(s) \tag{8} $$

In order to perfectly realize the fractional order operators, all the past inputs need to be memorized, which is impossible in real applications. There are mainly two discretization approaches for the approximation of the operators s^α, direct discretization and indirect discretization [27]. Various direct discretization methods have been proposed, such as short memory principle, sampling time scaling, and expansion of various operators such as Tustin, Al-Alaoui, and Euler operators by power series expansion (PSE) or CFE [28,29]. For the PSE method, the differential equations are in FIR filter structure while the approximation equations for the CFE method are in IIR filter structure. It has been shown that the CFE method is more efficient than the PSE method since the low order approximation equations with IIR structure can have excellent approximations, which can only be achieved by the FIR structure with high order equations [30].

In this paper, the CFE expansion of Euler operator is introduced, namely,

$$ s^\alpha = \left(\frac{1-z^{-1}}{T}\right)^\alpha \tag{9} $$

After CFE expansion, the discretization result is as the following equation:

$$ Z\{D^\alpha x(t)\} = \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{T}\right)^\alpha\right\}X(z) \approx \left(\frac{1}{T}\right)^\alpha \frac{P_p(z^{-1})}{Q_q(z^{-1})}X(z) \tag{10} $$

where $P_p(z^{-1})$ and $Q_q(z^{-1})$ are the polynomials with the orders of p and q, respectively. Usually, p and q can be set to be equal, $p = q$. The experimental results show that the tenth-order approximation of Euler operator is usually sufficient for engineering applications [31]. Therefore, in the following numerical analysis, the orders of p and q here are chosen as ten.

3 Fractional Damped Duffing System

The Duffing equation, a well-known nonlinear second-order differential equation, is used to describe many physical, engineering, and even biological problems [32]. The equation is given by

$$ m\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + kx(t) + \lambda x^3(t) = A\sin(\omega t) \tag{11} $$

where m, c, k, λ, A, and ω are mass, damping coefficient, linear stiffness, nonlinear stiffness, excitation amplitude, and excitation frequency, respectively. In conventional Duffing equation, the damping force is proportional to the first-order derivative of the displacement $x(t)$. However, many successful applications in mechanical engineering have been reported by expanding of the integer order damping to a fractional order one because fractional order damping can describe the complicated frequency dependency of damping materials [17–19]. The fractional order damping force is

$$ F_d = cD^\alpha x(t) \tag{12} $$

where α is the fractional order of the damping. Therefore, Eq. (11) can be rewritten as

$$ m\frac{d^2}{dt^2}x(t) + cD^\alpha x(t) + kx(t) + \lambda x^3(t) = A\sin(\omega t) \tag{13} $$

With the following property of sequential fractional derivatives [26]

$$ \begin{gathered} D^\alpha x(t) = D^{\alpha_1}D^{\alpha_2}, \ldots, D^{\alpha_{n-1}}D^{\alpha_n}x(t) \\ \alpha = \alpha_1 + \alpha_2 + , \ldots, \alpha_{n-1} + \alpha_n \end{gathered} \tag{14} $$

and zero initial value condition, Eq. (13) can be transformed into state equations, which are given by

$$ \begin{gathered} \frac{d^\alpha x}{dt^\alpha} = y \\ \frac{d^{1-\alpha}y}{dt^{1-\alpha}} = z \\ \frac{dz}{dt} = \frac{1}{m}(A\sin(\omega t) - \lambda x^3 - kx - cy) \end{gathered} \tag{15} $$

The first two fractional order derivative equations in Eq. (15) can be simulated using the CFE expansion of Euler operator, as shown in Eq. (10). The third equation can be numerically computed by the Runge–Kutta method. The experiments demonstrated that the fourth-order was adequate.

4 Results and Discussions

Dynamic trajectory can be used to check whether the system is periodic or nonperiodic. However, it cannot provide enough information to determine the onset of chaotic motion. Other analytical methods are necessary such as bifurcation diagram, phase diagram, Poincaré map, and Lyapunov exponent. The points on the Poincaré map represent the return points for a time series with constant interval T, where T is the driving period of the exciting force. For quasi-periodic motion, the return points in the Poincaré map form a closed curve. While for chaotic motion, the return points in the Poincaré map form a geometrically fractal structure. As to a periodic motion, the n discrete points in the Poincaré map indicate that the period of motion is nT. The Poincaré map can better identify the motion behavior of system with given parameters. At the same time, system dynamics with a range of parameter variation can be observed thoroughly using bifurcation diagrams [33]. The bifurcation diagram can provide valuable insights into system's nonlinear dynamic behavior. In this paper, the bifurcation diagrams are plotted under parameter variations with constant interval. The dynamic behaviors of fractional order damping

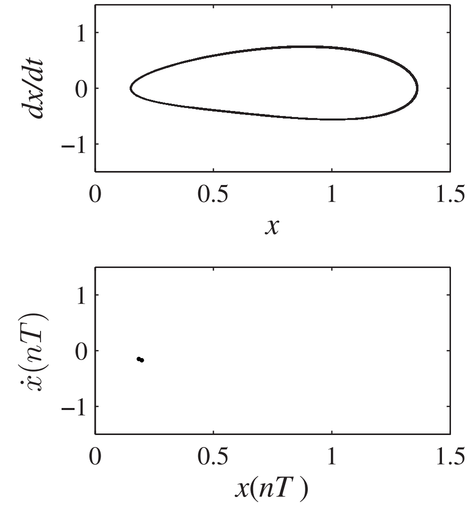

Fig. 1 Phase trajectory and Poincaré map at α=1.0

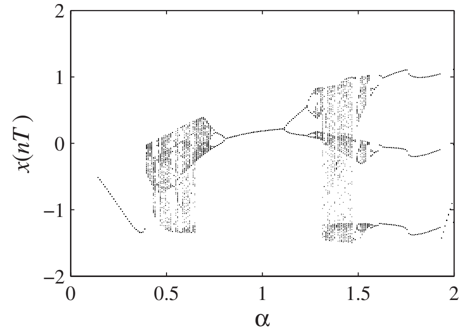

Fig. 2 Bifurcation diagrams of *x* versus α

Duffing are analyzed with various fractional order of damping, excitation frequency and amplitude and damping coefficient.

In this paper, the nonlinear dynamics of Duffing system with fractional order damping is simulated using MATLAB/SIMULINK. The fractional derivatives of Eq. (15) are approximated using the CFE expansion of Euler operator in which the order of the approximation is 10. First, fixed parameters $m=1$, $\lambda=1$, $k=-1$, $c=0.9$, $\omega=1$, and $A=0.6$ are adopted. And zero initial condition is selected, i.e., $x(0)=0$, $y(0)=0$, and $z(0)=0$.

In order to test the approximation method for numerical simulation, the case of $\alpha=1.0$ is calculated. When $\alpha=1.0$, the system is actually described by the conventional Duffing equation with integer order damping. Its phase trajectory and Poincaré map are shown using solid line and point in Fig. 1 as a baseline for comparison. Next, the above conventional Duffing system with the same parameters is simulated again but using the proposed approximation method for the fractional order derivatives, as shown in Eq. (10). The new results are plotted using dashed-dotted line and point, as show in Fig. 1. A good agreement can be observed between solid and dashed-dotted lines and points in the figure. The average square error from the baseline by using the approximation method is 0.00152625, which verifies its accuracy for simulating the fractional order damped Duffing system.

4.1 Influence of Fractional Order Damping. The fractional order varies from 0.08 to 2.0. Bifurcation can be easily detected by examining the relationship between x and the fractional order α. The bifurcation diagram with step size of $\Delta\alpha=0.005$ is shown in Fig. 2. At each value of the fractional order α, the first 50 points of the Poincaré map are discarded and the values of x for next 100 points are plotted in the bifurcation diagram. It can be observed that the fractional order significantly affects dynamic characteristics. When $0.08<\alpha\leq 0.387$, the response of Duffing system with fractional order damping is a periodic motion. As shown in Fig. 3(*a*) for $\alpha=0.38$, there is one isolated point in the Poincaré map and the phase trajectory shows a regular period-1 motion. After

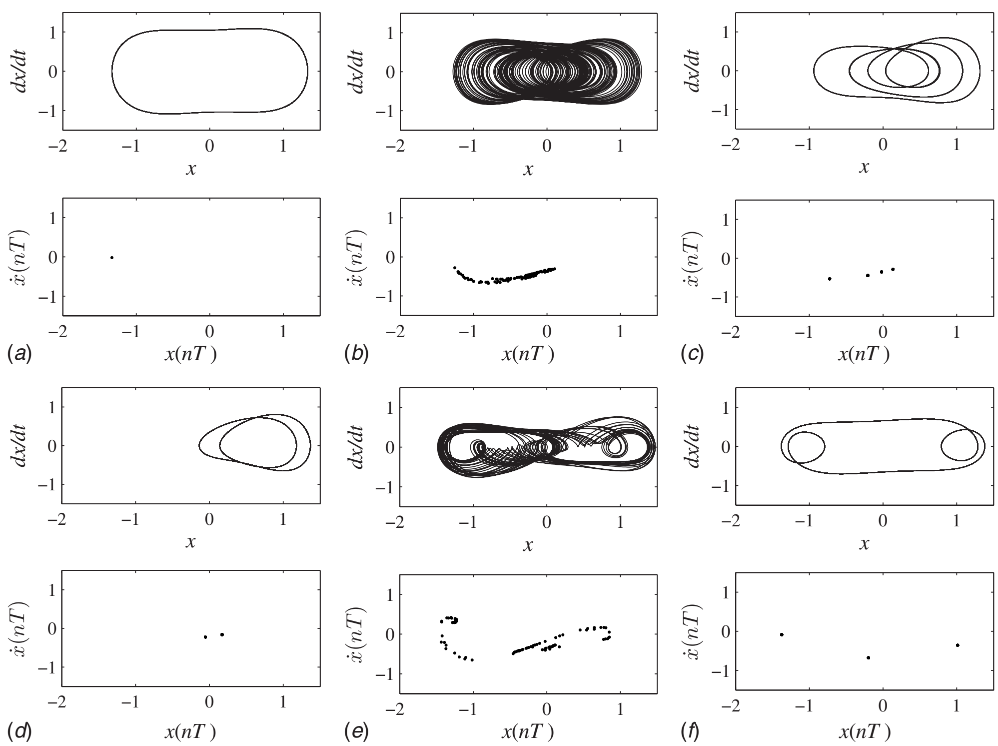

Fig. 3 Phase trajectory and Poincaré map with various α

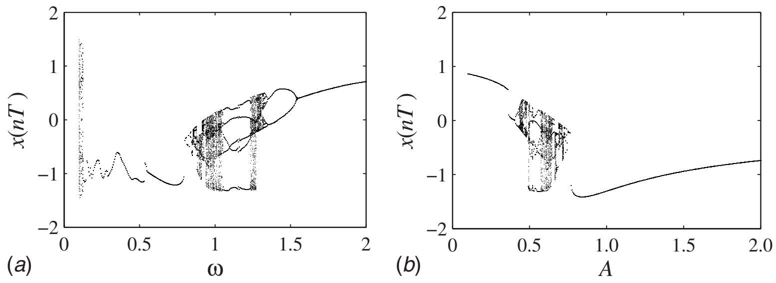

Fig. 4 Bifurcation diagrams with various ω and A

undergoing the periodic motion zone, the motion suddenly comes into the first chaotic region. Hence the chaotic state remains from 0.388 to 0.733.

Figure 3(*b*) shows the phase trajectory and Poincaré map for $\alpha=0.48$. There is a strange attractor representing chaotic motion in the Poincaré map. And the phase trajectory shows an irregular motion. In order to clearly identify the dynamic behavior from a quantitative viewpoint, the largest Lyapunov exponent developed by Wolf et al. [34] is introduced to explain the characteristics of system behavior. The corresponding largest Lyapunov exponent when $\alpha=0.48$ is 1.0596. In addition, the periodic motion windows appear in the first chaotic motion zone. For $\alpha=0.50$. a period-4 motion can be identified from the four isolated points in the Poincaré map (see Fig. 3(*c*)). If the fractional order α further increases, the system response returns to periodic motion. When $\alpha=0.7336$, the motion is a period-32 and then becomes period-1 when $\alpha=0.81$ by an inverse period doubling bifurcation. Figure 3(*d*) shows the phase trajectory and Poincaré map for $\alpha=0.75$. It can be seen that the periodic window is a period-2 motion.

When $\alpha>1.1$, the system response gradually enters into the second chaotic zone by the route of period doubling bifurcation. The second chaotic zone for α is from 1.28 to 1.58. As shown in Fig. 3(*e*) when $\alpha=1.38$, again there is a strange attractor showing chaotic motion in Poincaré map and the corresponding largest Lyapunov exponent is 1.0752. With further increase in α when $\alpha>1.58$, the motion returns to the periodic motion region. Figure 3(*f*) with $\alpha=1.78$ clearly exhibits a period-3 motion.

From the above analysis, it can be concluded that when $0.08<\alpha<2.0$ the fractional order damped Duffing system exhibits the periodic, chaotic, periodic, chaotic, and periodic motion in turn. The motion turns into chaos through a route of sudden transition from the periodic to chaotic motion when $0.1<\alpha<0.75$ and then leaves chaos by an inverse period doubling bifurcation. When $\alpha>1.1$, it comes into chaos again through a route of period doubling bifurcation and leaves chaos through a route of period reducing bifurcation. The system dynamics eventually becomes a period-3 motion. In addition, the periodic motion windows appear in the both chaotic motion zones.

4.2 Influences of Excitation Frequency and Amplitude. The above analysis and conclusion focus on the effect of fractional order α on system's dynamic behavior. However, the excitation frequency and amplitude always play an important role in dynamic characteristics. In the analysis below, the fractional order α is fixed to 0.5 while the excitation frequency ω and amplitude A are used as control parameters. The bifurcation diagram with various ω is shown in Fig. 4(*a*), where $\alpha=0.5$, $A=0.6$, $c=0.9$, and the bifurcation diagram with different A is shown in Fig. 4(*b*) with $\alpha=0.5$, $\omega=1.0$, and $c=0.9$. The constant interval for the variations of ω and A in the below two bifurcation diagrams is set as 0.005.

In Fig. 4(*a*), the excitation frequency varies from 0.1 to 2.0. When $0.2<\omega<0.85$, the fractional order damped Duffing system's response is a periodic motion. As shown in Fig. 5(*a*), when $\omega=0.75$, there is one isolated point in the Poincaré section and the phase trajectory shows a regular period-1 motion. After the periodic motion zone, the motion gradually enters the chaotic region by the route of period doubling bifurcation. The chaotic state remains from 0.94 to 1.32. When $\omega=1.26$, a strange attractor appears representing chaotic motion in the Poincaré map and the phase trajectory shows an irregular motion (see Fig. 5(*b*)). In addition, the large periodic motion windows from $\omega=1.05$ to 1.24 appear in the chaotic motion zone. As shown in Fig. 5(*c*) for $\omega=1.12$, a period-5 motion can be identified from the five isolated points in the Poincaré map. When ω is further increased, the system response returns to period-1 motion through the route of inverse period doubling bifurcation.

It can be seen from Fig. 4(*b*) that when the excitation amplitude A increases from 0.1 to 2, the system response comes into the chaotic zone by the route of period doubling bifurcation, and leaves the chaotic zone by the route of inverse period doubling

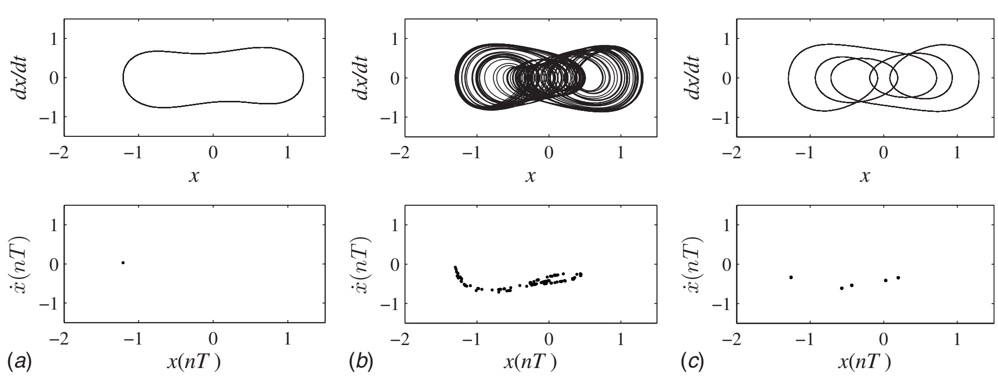

Fig. 5 Phase trajectory and Poincaré map with various ω

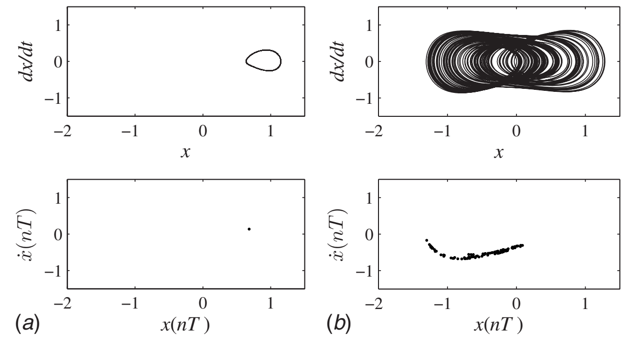

Fig. 6 Phase trajectory and Poincaré map with various *A*

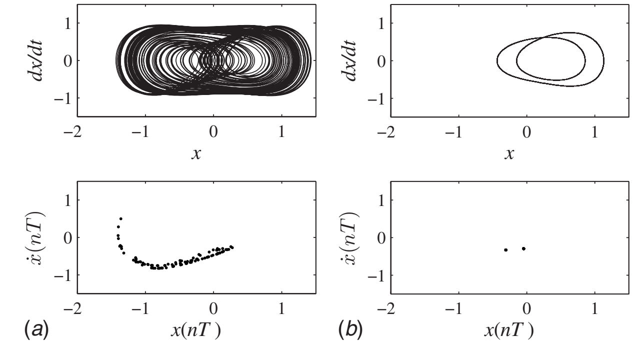

Fig. 8 Phase trajectory and Poincaré map with various *c*

bifurcation. When $0.1 < A < 0.403$, the fractional order damped Duffing system's response is a period-1 motion as illustrated in Fig. 6(*a*). There is one isolated point in the Poincaré map and the phase trajectory shows a regular period-1. When the excitation amplitude A increases from 0.403, the system response gradually enters the chaotic zone through the period doubling bifurcation. It is clear that when $A=0.42$, the motion is a period-2 and then becomes a period-4 for $A=0.43$. Figure 6(*b*) shows the phase trajectory and Poincaré map for $A=0.65$ as an example of further increase in A. A strange attractor can be observed, which is representing chaotic motion in the Poincaré section and the trajectory shows an irregular motion. From Fig. 4(*b*), it can be seen that periodic motion windows appear in the chaotic region. When $A > 0.75$, the system response gradually returns to periodic motion zone through the inverse period doubling bifurcation.

4.3 Influence of Damping Coefficient. The damping coefficient is one of the important factors for adjusting the system dynamics in many engineering applications. For example, the vibration of rotating machinery is always suppressed through the change in damping. It is necessary to analyze the effect of damping coefficient on the dynamic of fractional order damped Duffing system. The damping coefficient c is used as control parameter for the bifurcation diagram in Fig. 7. The other parameters for this case are $\alpha=0.5$, $A=0.6$, $\omega=1.0$, and the step size of the control parameter c is 0.005. It can be seen from Fig. 7 that as the damping coefficient increases from 0.1 to 2.0, the system motion suddenly enters into the chaos from the period-1 and then leaves chaos by the route of inverse period doubling bifurcation. Figure 8(*a*) shows the phase trajectory and Poincaré map for $c=0.17$. There is a strange attractor representing chaotic motion in the Poincaré map and the trajectory shows an irregular motion. The inverse period doubling bifurcation is clear in Fig. 7 and, finally, the system return to the periodic motion. Figure 8(*b*) shows the phase trajectory and Poincaré map for $c=1.13$. There are only two points in the Poincaré map. Therefore, the system response is period-2 motion.

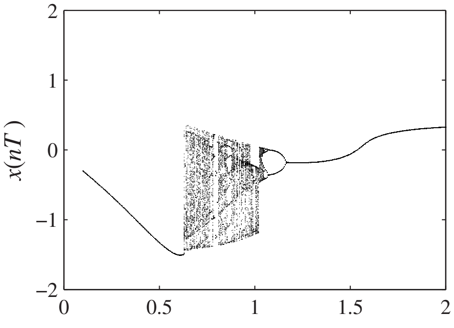

Fig. 7 Bifurcation diagrams of *x* versus *c*

5 Conclusions

The nonlinear dynamics of the fractional order damped Duffing system is investigated in this paper. The fourth-order Runge–Kutta method and tenth-order CFE-Euler approximation method are adopted to simulate the fractional order damped Duffing equations. The numerical simulation results with $\alpha=1.0$ show the CFE-Euler approximation method is proper for approximating the fractional order equations.

The phase diagram, the Poincaré diagram, the bifurcation diagram, and the largest Lyapunov exponent are introduced to evaluate the effect of the fractional order damping on dynamic behaviors. The analysis shows that the fractional order damped Duffing system exhibits periodic motion, chaos, periodic motion, chaos, and periodic motion in turn when the fractional order changes from 0.1 to 2.0. A period doubling route to chaos and inverse period doubling route from chaos to periodic motion can be clearly observed. The bifurcation diagram is introduced to investigate the effects of excitation amplitude, frequency, and damping coefficient on the Duffing system with fractional order damping. It is observed that the fractional order damped system exhibits the complicated nonlinear dynamic behavior under external excitation.

The numerical results verify the significant effect of fractional order damping on system dynamics. Therefore more attention should be paid to the damping with fractional order for the design, analysis and control of system dynamics. Specifically, the dynamic analysis of rotor bearing system is important for the exact diagnosis of malfunctions and improving the dynamic characteristics. The further research would introduce the concept of fractional order damping to analyze the nonlinear behavior of rotating machinery and, thus, enhance the dynamic analysis accuracy and maintenance efficiency.

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