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NONLINEAR DYNAMICS OF DUFFING SYSTEM WITH FRACTIONAL ORDER DAMPING

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ABSTRACT
In this paper, nonlinear dynamics of Duffing system with fractional order damping is investigated. The four order Runge-Kutta method and ten order CFE-Euler methods are introduced to simulate the fractional order Duffing equations. The effect of taking fractional order on the system dynamics is investigated using phase diagrams, bifurcation diagrams and Poincare map. The bifurcation diagram is also used to exam the effects of excitation amplitude and frequency on Duffing system with fractional order damping. The analysis results show that the fractional order damped Duffing system exhibits period motion, chaos, period motion, chaos, period motion in turn when the fractional order changes from 0.1 to 2.0. A period doubling route to chaos is clearly observed.

1. INTRODUCTION
Fractional Calculus is a branch of applied mathematics that studies the possibility of taking arbitrary orders of the differential and integration operators. The applications of fractional caculus in engineering and physics have attracted lots of attention. Because fractional calculus is having profound impact on many engineering and scientific areas such as signal and image processing, mechanics, mechatronics, physics, control theory, viscoelasticity and rheology, electrical engineering, electrochemistry and bioengineering[1-2]. Although some of the mathematical issues remain unsolved, fractional calculus based modeling of complicated dynamics is becoming a recent focus of interest. The dynamics of fractional order system equations for Chua, Lorenz, Rossler, Chen, Jerk and Duffing are mainly investigated [3-9]. It is obvious that the chaotic attractors existing in their fractional systems have different fractional orders. In Ref. 10，Long-Jye Sheu et al mainly researched the effect of fractional order damping on dynamic behaviors. In Ref. 11 and Ref. 12, bifurcation and chaotic dynamics of the fractional order cellular neural networks were studied. The fractional Van der Pol equation with periodically exciting was investigated in Ref. 13. It has been shown that the chaotic motions exist when the order of fractional damping is less than 1.

In recent years, the dynamics and vibration analysis of fractional order damped systems are of great interest to researchers[14-19]. The fractional order operator’s characterstic of having an unlimited memory leads to a concise description of complicated system dynamics. For example, the backlash and impact can be more adequately analyized [20-21]. Zheng-Ming Ge et al introduced the chaos control of the fractional order rotational mechanical system[22]. Machado et al also explained that while the dynamics of each individual element has an integer-order nature, the global dynamics reveals the existence of both integer and fractional order nature [23]. Therefore, it is essential to consider the fractional order damping in study the dynamic characteristics.

The Duffing equation, which is being used in many physical, mechanical and even biological engineering problems, has been modified to study the dynamics of fractional order systems[4,5,10]. Although much work has been done on the chaotic dynamics of fractional order systems, the preceding researches mainly focused on the effect of the fractional order damping. There are rare papers investigating the effect of other parameters including the amplitude and frequency of the external exciting force. Because these parameters also play an important role in the dynamic characteristics of fractional order system, it is necessary to study the impact of the above parameters on the fractional dynamics. Therefore, this article discusses the nonlinear analysis of fractionally damped Duffing with the variation of not only the fractional order, but also the amplitude and frequency of the external exciting force. It is well-known that the fractional differintegral operators do not allow direct implementation in time-domain simulations. Appropriate approximations of fractional operator need to be developed for the analysis. There is significant interest in developing numerical methods for simulating fractional differantial equations. In Ref.3 to Ref.6, a linear approximation of fractional order transfer function in frequency domain is adopted to study the chaotic characteristics. In Ref. 10 and Ref. 13, a predictor–corrector approach with numerical schemes for Volterra integral equation is proposed. The direct approximation using Euler rule and Continued Fractional Expansion will be implemented for numerial simulation of fractional Duffing system in this paper. The proposed approximation method is preferable for physical engineering applications.

2. FRACTIONAL CACULUS AND DISCRETIZATION SCHEMES

There are two definitions for fractional differentiation and integration, the Grunwald-Letnikov (GL) definition and the Riemann–Liouville (RL) definition [24]. The GL definition is the best known one since it is most direct for the digital realization of the fractional order operators. The GL fractional derivative of continuous function $f(t)$ is given by:

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[x]} (-1)^{-j} \binom{\alpha}{j} f(t - jh) \tag{1}$$

where $[x]$ is a truncation and $x = \frac{t-\alpha}{h}$. $\binom{\alpha}{j}$ is binomial coefficients, it can be replaced by the Gamma function, $\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)}$. While the RL definition is given by:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d(\tau) \tag{2}$$

for $(n-1 < \alpha < n)$

In Laplace domain it is usually more easily to describe the fractional integro-differential operation. The Laplace transform of the fractional integral of is given by:

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s) \tag{3}$$

where $F(s)$ is the Laplace transform of $f(t)$. The Laplace transform of the fractional derivative of $f(t)$ is given as follows:

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s) - \sum_{k=1}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1} f(0) \right] \tag{4}$$

where $n-1 < \alpha < n$ again. If all the initial conditions are zeros, the Laplace transform of fractional derivative is simplified to Eqn.(3).

In order to exactly realize the fractional order operators, all the past input need to be memorized, which is not practical. These are mainly two discretization methods to approximate the operator s^{α} . They are direct discretization and indirect discretization[25]. Several direct discretization methods by finite differential equation have already been proposed, such as the Short memory principle, Tustin expansion, Al-Alaoui expansion [26]. In order to calculate the coefficients of the approximated differential equations, Power Series Expansion (PSE) and Continued Fraction Expansion (CFE) can be introduced. For the PSE method, the differential equations are in FIR filter structure; while as to the CFE method, the approximation equations are in IIR filter structure. It has been shown that the low order approximation equations with IIR structure can achieve excellent approximations, which can only be achieved by the FIR structure with high order [27]. Namely the CFE method is more efficient than the PSE method. The experimental results also show that the ten order approximation equation with Euler and Al-Alaoui are proper for engineering applications [28]. The following steps explain is the adoptation of Euler operator for direct discretization of the fractional order operator, which can be given by:

$$ s^{\alpha} = \left(\frac{1 - z^{-1}}{T} \right)^{\alpha} \tag{5}$$

Then perform CFE, the discretization result is as follows:

$$ \begin{aligned} Z\left\{ D^{\alpha} x(t) \right\} &= CFE\left\{ \left(\frac{1-z^{-1}}{T} \right)^{\alpha} \right\} X(z) \\ &\approx \left(\frac{1}{T} \right)^{\alpha} \frac{P_p(z^{-1})}{Q_q(z^{-1})} X(z) \end{aligned} \tag{6}$$

where $Z\{u\}$ denotes the z-transformation of u and $CFE\{u\}$ denotes the Continued Fraction Expansion of *u*; *p* and *q* are the orders of the approximation; *P* and *Q* are the polynomials of degrees *p* and *q*. Usually *p*, *q* and *n* can be set to be equal, $p = q = n$. In the below numerical analysis, the order of the approximation equation is chosen as 10.

3. FRACTIONAL DAMPED DUFFING

The Duffing equation, a well-known nonlinear differential equation, is used for decsribing many physical, engineering and even biological problems[29]. Originally the Duffing equaiton was introduced by German electrical engineer Duffing in 1918. The equation is given by:

$$m\frac{d^2}{dt^2}x(t)+c\frac{d}{dt}x(t)+kx(t)+\lambda x^3(t)=A\sin(\omega t) \quad (7)$$

Where m, c, k, λ, A and ω are the mass, damping coefficient, linear stiffness, nonlinear stiffness, excitation amplitude and excitation frequency, respectively. The different values of λ show the hard and soft characteristics of spring. When $k=-1$, the conventional Duffing equation will be the famous Holmes type Duffing oscillator. It is can be seen that in Duffing equation the damping force is proportional to the one order derivatives of the displacement. The damping modeling using fractional derivative has many successful applications in the mechanical engineering[30-33], because it can describe the complicated frequency dependency of damping materials. The fractional order damping force is described by:

$$F_d = cD^{\alpha}x(t) \quad (8)$$

where α is the fractionalorder of damping. The Eqn. (7) and Eqn. (8) can be integrated into

$$m\frac{d^2}{dt^2}x(t)+cD^{\alpha}x(t)+kx(t)+\lambda x^3(t)=A\sin(\omega t) \quad (9)$$

Based on the following property of sequential fractional derivatives[24]

$$\begin{aligned} &D^{\alpha}x(t)=D^{\alpha_1}D^{\alpha_2}\cdots D^{\alpha_{n-1}}D^{\alpha_n}x(t) \\ &\alpha=\alpha_1+\alpha_2+\cdots+\alpha_{n-1}+\alpha_n \end{aligned} \quad (10)$$

and zero initial value condition, The Eqn. (9) can be transformed into the state equations, which is given by:

$$\begin{cases} \dfrac{d^{\alpha}x}{dt^{\alpha}}=y \\ \dfrac{d^{(1-\alpha)}y}{dt^{(1-\alpha)}}=z \\ \dfrac{dz}{dt}=\dfrac{1}{m}(f\sin\omega t-\lambda x^3-kx-cy) \end{cases} \quad (11)$$

The first two fractional derivative equations in Eqn. (11) can be simulated using the approximation method shown in Eqn. (6). The third equation can be numerically computated by four order Runge–Kutta method.

4. RESULTS AND DISCUSSIONS

The dynamic trajectories can be used to distinguish whether the system is periodic or non-periodic. However, it cannot provide enough information to determine the onset for chaotic motion. Therefore, it is necessary to introduce other analytical methods, which are bifurcation diagram, phase diagram, Poincare map and Lyapunov exponents. The points on the Poincare map represent the return points of the time series at a constant interval T, where T is the driving period of the exciting force. For quasi-periodic motion, the return points in the Poincare map form a closed curve. For chaotic motion, the return points in the Poincare map form a particular structure or a geometrically structure. As to a periodic motion, the n discrete points on the Poincare map indicate that the period of motion is nT. It is clear that the Poincare map can better identify the motion behavior of system with given parameters[34]. However, it is necessary that the system dynamics with a range of parameter values need to be viewed thoroughly using bifurcation diagrams. Because the bifurcation diagrams can summarize the essential dynamics of system, therefore provide valuable insights into its nonlinear dynamic behavior. In this paper, the bifurcation diagrams are generated by parameter variation with a constant step, and the emphasis is on analysis the effects of fractional order of damping, excitation frequency and excitation amplitude. The above methods will be adopted to analyze various dynamic behaviors of fractional damping Duffing.

The nonlinear dynamics of Duffing system with fractional damping is digitally simulated in Matlab/Simulink. The fractional derivatives of Eqn. (11) is approximated by the IIR discrete model using CFE and Euler rule, in which the order of approximation model is set to be 10. In this study, we firstly fix parameters $m=\lambda=\omega=1$, $k=-1$ $c=0.9$ and $f=0.6$ .The initial state is set to $x(0)=0, y(0)=0, z(0)=0$. In order to test the numerical scheme, the case with $\alpha=1.0$ is calculated. When $\alpha=1.0$, the system is actually described by the classical Duffing equation. The phase portrait and Poncare map are shown in Fig.1. The classical Duffing system with the same parameters using Runge–Kutta method is simulated. The results are in good agreement with Fig. 1. And when $\alpha=1.0$, the average square error of each step output between the proposed numerical scheme and Runge–Kutta method is 0.00152625. Therefore, the proposed numerical scheme is verified to be accurate for simulating the fractional Duffing system. The stable response can be obtained by discarding the output of former 50 excitation periods and retaining the output of last 100 excitation periods. Since the sample frequency in numerical computation is 100 times of the excitation frequency, the number of data points plotted in diagrams is 10000.

The effect of the fractional α order damping on the dynamic behavior of the system is mainly investigated. The fractional order ranges from 0.08 to 2. Bifurcation can be easily detected by examining the relationship between x and the fractional order α . The bifurcation diagram with step size $\Delta\alpha=0.005$ is shown in Fig.2. At each value of the fractional order α , the first 50 points of the Poincare map are discarded and the values of x for next 100 points are plotted on the bifurcation diagram. From Fig. 2, it can be observed that the fractional order exhibits a significant effect on dynamic characteristics. When $0.08<\alpha\le 0.387$, the response of Duffing system with fractional order damping is a period motion. Fig.3 shows the

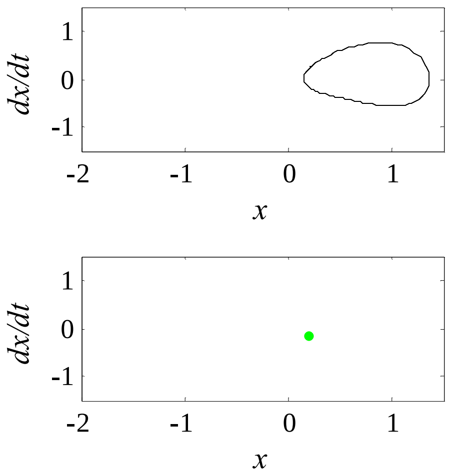

Figure 1. Phase trajectory and Poincare map at α =1.0.

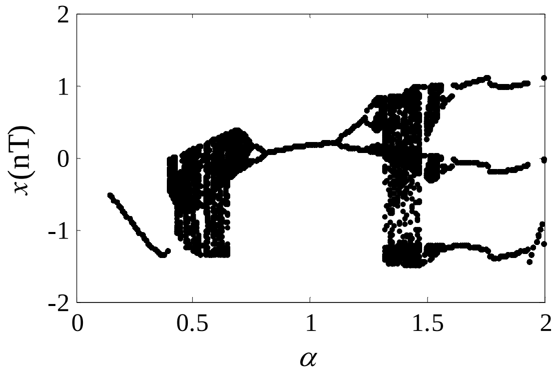

Figure 2. Bifurcation diagrams of x versus α

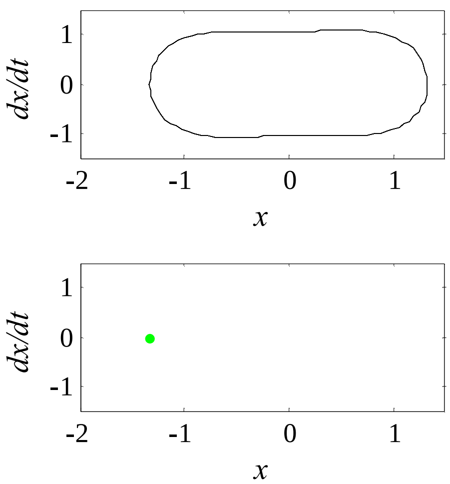

Figure 3. Phase trajectory and Poincare map at α =0.38.

significant effect on dynamic characteristics. When $0.08 < \alpha \le 0.387$, the response of Duffing system with fractional order damping is a period motion. Fig.3 shows the phase trajectory and Poincare map for $\alpha = 0.38$. There is one isolated point in the Poincare section and the trajectory shows a regular period one. After undergoing the period motion zone, the motion suddenly comes into the first chaotic region. Hence the chaotic state remains from 0.388 to 0.733.

Fig.4 shows the phase trajectory and Poincare map for $\alpha = 0.48$. There is a strange attractor representing chaotic motion in the Poincare section. And the trajectory shows an irregular motion. In order to clearly identify the dynamic behavior from a quantitative view of point, the largest Lyapunov exponent is introduced to explain the characteristics of system behavior. The corresponding largest Lyapunov exponent at $\alpha = 0.48$ is 1.0596. However, the period motion windows appear in the first chaotic motion zone. Fig.5 shows the phase trajectory and Poincare map for $\alpha = 0.50$. It is obvious that the period four motion can be identified from the four isolated points in the Poincare section. When the fractional order further increases, the system response returns to period motion. When $\alpha = 0.75$, the motion is a period two and then becomes period one when $\alpha = 0.81$ by an inverse period doubling bifurcation. Fig.6 shows the phase trajectory and Poincare map for $\alpha = 0.75$. The periodic window is identified as period-2 motion from Fig. 6.

when $\alpha > 1.1$, the system response gradually enters into the second chaotic zone by the route of period doubling bifurcation. The second chaotic zone is from 1.28 to 1.58. Fig.7 shows the phase trajectory and Poincare map for $\alpha = 1.38$. Again there is a strange attractor showing chaotic motion in Poincare section and the corresponding largest Lyapunov exponent at $\alpha = 1.38$ is 1.0752. Finally, the motion again comes into the period motion region along with the further increased fractional order from $\alpha > 1.58$. Fig. 8 shows the phase trajectory and Poincare map at $\alpha = 1.78$, which also exhibits clearly a period-3 motion.

It can be concluded from the above analysis, when $0.08 < \alpha < 2.0$, the fractional order damped Duffing system exhibits the periodic, chaotic, periodic, chaotic, periodic motion in turn. At the same time, the motion turns into chaos through a route of sudden transition from the periodic to chaotic motion when $0.1 < \alpha < 0.75$, and then leaves chaos by an inverse period doubling bifurcation. When $\alpha > 1.1$, it comes into chaos again through a route of period doubling bifurcation and leaves chaos through a route of period reducing bifurcation. Finally, the system dynamics becomes a period-3 motion. In addition, the periodic motion window appears in the both chaotic motion zone.

The above analysis and conclusion mainly focus on the effect of fractional order α on system's dynamic behavior. However, the excitation frequency and amplitude always play a significant role on dynamic characteristics. In the following analysis, the fractional order α is set to be constant and the excitation frequency ω and amplitude A are used as control parameters.

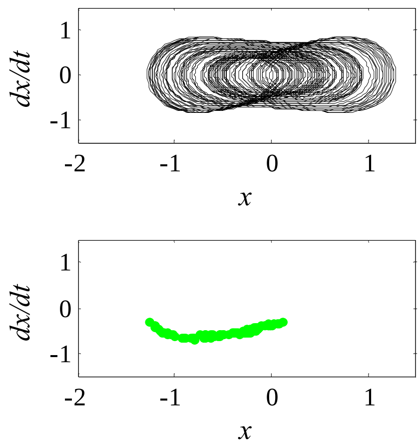

Figure 4. Phase trajectory and Poincare map at α =0.48.

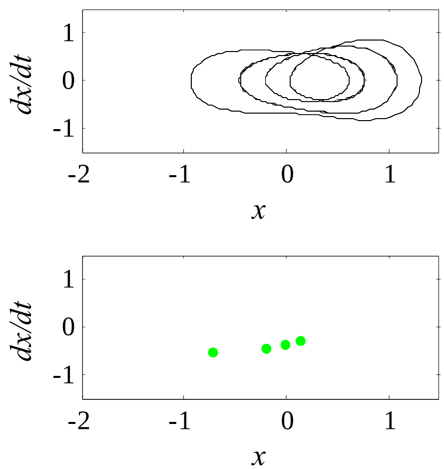

Figure 5. Phase trajectory and Poincare map at α =0.5.

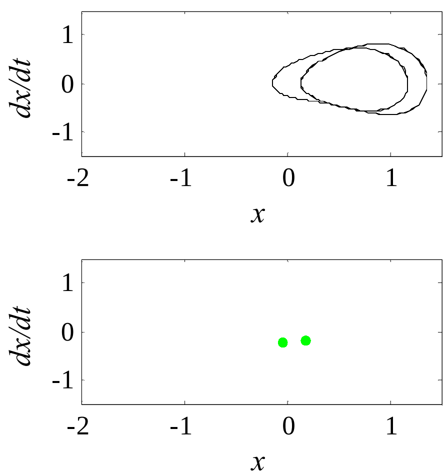

Figure 6. Phase trajectory and Poincare map at α =0.75.

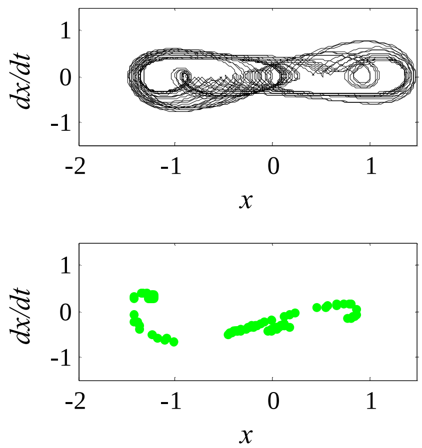

Figure 7. Phase trajectory and Poincare map at α =1.38.

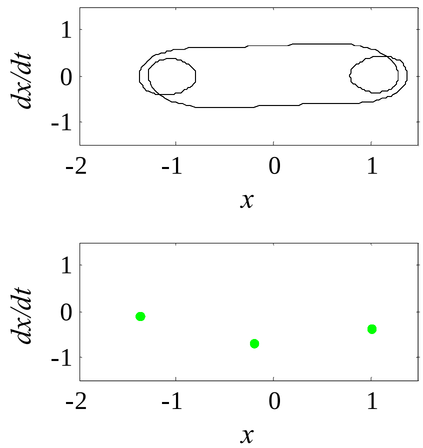

Figure 8. Phase trajectory and Poincare map at α =1.78.

The bifurcation diagram with various control parameter ω is shown in Fig.9 where $\alpha = 0.5, A = 0.6, c = 0.9$ .While the bifurcation diagram with different control parameter A is shown in Fig. 10 where $\alpha = 0.5, \omega = 1.0, c = 0.9$. Clearly the fractional order damped system exhibits the complex nonlinear dynamic behavior under the external excitation.

5. CONCLUSION

The nonlinear dynamics of the fractionally damped Duffing system are investigated in this paper. The four order Runge-Kutta method and ten order CFE-Euler approximation methods are combined to simulate the fractional order Duffing equations. The numerical simulation results when $\alpha = 1.0$ show the CFE-Euler approximation methods is proper for approximating the fractional order equations.

The phase diagram, Poincare diagram, bifurcation diagram and the largest Lyapunov exponents are introduced to

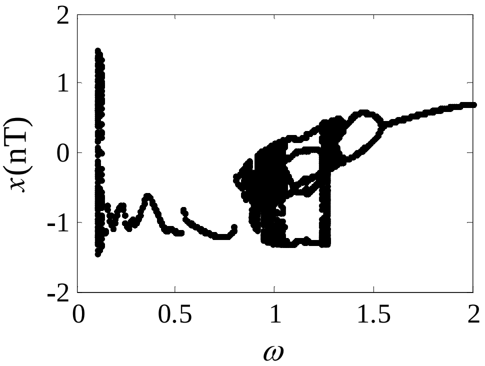

Figure 9. Bifurcation diagrams of x versus ω

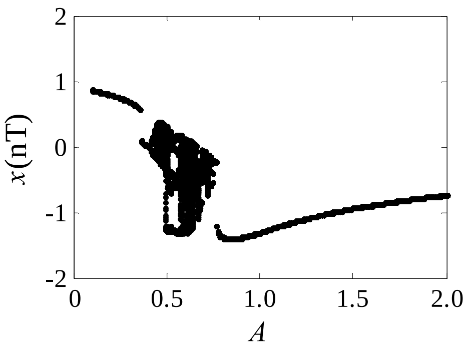

Figure 10. Bifurcation diagrams of x versus A.

evaluate the effect of the fractional order of damping on dynamic behaviors. The analysis shows that the fractional order damped Duffing system exhibits period motion, chaos, period motion, chaos, period motion in turn when the fractional order changes from 0.1 to 2.0. A period doubling route to chaos and inverse period doubling route from chaos to periodic motion can be clearly observed. The bifurcation diagram is also introduced to investigate the effect of excitation amplitude and frequency on the Duffing system with fractional order damping.

The numerical results verify the significant effect of fractional order on system dynamics. Therefore more attention should be paid to the fractional order of damping for the design, analysis and control of system dynamics. More specifically, the dynamic analysis of rotor bearing system is important for exactly diagnosing the malfunctions and improving the dynamic characteristics. The further research would introduce the concept of the fractional order damping to analyze the nonlinear behavior of rotating machinery and thus enhance the dynamic analysis accuracy and the maintenance efficiency.

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