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Optimized Characteristic Ratio Assignment for Low-Order Controller Design

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***Abstract*—It is known that polynomial method is particularly suitable for designing low-order controllers. So far most of the controller design using the polynomial method was based on a pre-defined standard form of the characteristic ratio assignment (CRA). Meanwhile, with limited parameters of the low-order controllers, an exact CRA following a standard form becomes impossible when the order of the control plant is sufficiently high. This paper proposes a systematic design scheme for an optimized CRA. First the influences of the characteristic ratios are quantified as weight coefficients. Then the objective function of the optimization problem is constructed to minimize the difference between the actual CRA and its nominal form. In addition to damping (i.e., the CRA), the requirements on the stability, speed of response, and robustness are also considered as constraints in an optimization problem. The so-called robust optimization problem is then formulated and solved via an inner-outer optimization formulation. Finally, the controller design for a three-mass benchmark system is applied a case study. The simulation results validate the propose scheme especially the robustness against parameter variation, unmodeled dynamics, and disturbance torque.**

***Index Terms*—Low-order controller design, polynomial method, characteristic ratio assignment, robust optimization, high-order systems.**

I. Introduction

Usually in controller design time responses serve as a final criterion for evaluating control performance. It is highly desirable to determine control parameters directly based on the transient characteristics such as overshoot and speed of response. In the so-called polynomial method, an algebraic design approach using polynomial expressions, controllers are designed via the assignment of characteristic ratios and generalized time constant [1]. Both the two types of parameters have clear physical meaning in time responses. The characteristic ratios have a strong relationship with damping, i.e., the overshoot, of a closed-loop system, while the generalized time constant relates to the speed of the response. The polynomial method can serve as an effective design tool for explicitly addressing the transient time response through proper assignments of the characteristic ratios and the generalized time constant. Besides, in its design procedure control configuration is first defined at the beginning. This makes the polynomial method especially suitable for designing low-order controllers such as the PID-based ones. It is both theoretically and practically important to further explore the possibilities of the polynomial method in solving real and complicated control problems.

The polynomial method was originally investigated by Kessler in 1960s, and later Naslin empirically observed the relationships between the characteristic ratios and the transient time responses [2], [3]. Manabe proposed and applied the Coefficient Diagram Method (CDM), which is based on Naslin's finding and Lipatov-Sokolov stability criterion [4]. Recent theoretical discussions include transient response control via characteristic ratio assignment (CRA), sensitivity analysis of time response to characteristic ratios, the assignment of the generalized time constant for non-all-pole systems, etc [1], [5]–[7]. Meanwhile, with limited parameters of the low-order controllers, an exact CRA following a standard form becomes impossible when the order of the control plant is sufficiently high. Moreover, in addition to the requirement on damping or overshoot, the influence of zeros, robustness, and stability need to be simultaneously considered for a practical and well-balanced controller.

This paper reports new results in the optimized CRA for the low-order controller design. The influences of the characteristic ratios are first quantified as weight coefficients. Then the objective function of the optimization problem is determined as to minimize the difference between the actual CRA and its standard/nominal form, i.e., the damping performance. In addition, the requirements on the stability, speed of response, and robustness are also treated as three constraints in the optimization problem. Particularly the speed of response, i.e., the assignment of the generalized time constant, is further discussed to achieve non-overshorting responses when zeros impose limitations. Moreover, a robust index is defined to represent and quantify the requirement on the robustness. Finally, this so-called robust optimization problem is formulated and solved via an inner-outer optimization approach. As a case study, the low-order controller design for a three-mass benchmark system is performed using the proposed scheme.

II. Preliminary Discussion

A. Definitions and Nominal CRA

For a control system with a closed-loop characteristic polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0, \qquad (1)$$

the characteristic ratios $\boldsymbol{\gamma}$ and generalized time constant τ in the polynomial method can be defined as

$$\boldsymbol{\gamma} = [\gamma_1,\ \gamma_2,\ ...,\ \gamma_{n-1}], \tag{2}$$

$$\gamma_1 = \frac{a_1^2}{a_0 a_2},\ \gamma_2 = \frac{a_2^2}{a_1 a_3},\ ...,\ \gamma_{n-1} = \frac{a_{n-1}^2}{a_{n-2} a_n},\ \tau = \frac{a_1}{a_0}. \tag{3}$$

Then the characteristic polynomial in (1) can be rewritten as a function of $\boldsymbol{\gamma}$ and τ,

$$D(s) = a_0 \left[\frac{1}{\gamma_{n-1}\gamma_{n-2}^2 \cdots \gamma_1^{n-1}} (\tau s)^n + ... + \frac{1}{\gamma_1}(\tau s)^2 + (\tau s) + 1 \right]. \tag{4}$$

For an all-pole system with the transfer function

$$G(s) = \frac{a_0}{D(s)} = \frac{a_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0}, \tag{5}$$

based on (4), the time response of $G(s)$ is scaled by the value of τ according to the Laplace transform of time-scaled functions, while the shape of the time response is solely determined by $\boldsymbol{\gamma}$. In addition, the lower-index characteristic ratios, especially γ_1, γ_2, and γ_3, have a more dominant influence [4], [6], [8]. Meanwhile, for non-all-pole systems, the existence of zeros may limit the feasible values of τ, namely an achievable speed of the response that does not lead to excessive overshoot/undershoot [1], [9]. For a nominal or target CRA here, it is known that overshoot can be adjusted by changing the single characteristic ratio γ_1 with all the other higher indexed characteristic ratios being fixed at 2 [1]. This specific CRA can be defined as

$$\gamma_1 = \gamma_1^* \text{ and } \gamma_i = 2 \text{ for } i = 2, 3, ..., n-1, \tag{6}$$

where γ_1^*'s are the minimum values of γ_1 that enable nonovershooting step responses. As listed in Table I, the values of γ_1^* are determined numerically by searching γ_1, while fixing all other higher indexed characteristic ratios at 2.

TABLE I
γ_1^*'S FOR NONOVERSHOOTING STEP RESPONSES

Order of System	3	4	5	6	7	8
γ_1^*	2.61	2.53	2.48	2.48	2.48	2.48

B. Quantification of Influences

The closed-loop transfer function in (5) can be rewritten as follows assuming a constant τ,

$$G(s,\ \boldsymbol{\gamma}) = \frac{1}{\frac{1}{\gamma_{n-1}\gamma_{n-2}^2 \cdots \gamma_1^{n-1}} (\tau s)^n + ... + \frac{1}{\gamma_1}(\tau s)^2 + \tau s + 1}. \tag{7}$$

Note for an all-pole system, the shape of its time response such as overshoot is solely determined by $\boldsymbol{\gamma}$. The influence of τ in non-all-pole systems will be discussed later in the following section as a constraint of the optimization problem. As listed in Table. I, a nominal CRA for high-order systems can be considered as

$$\boldsymbol{\gamma}^* = [2.48,\ 2,\ 2, ...]. \tag{8}$$

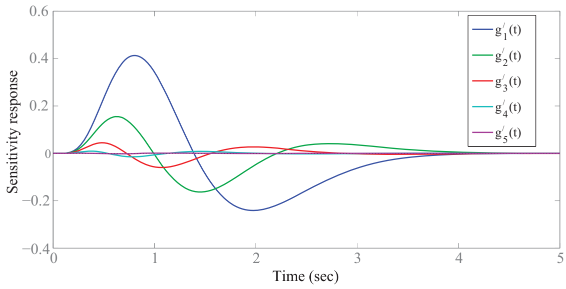

Fig. 1. Example sensitivity responses $g_i^{'}(t)$ for a sixth-order all-pole system.

Let $\Delta\boldsymbol{\gamma}$ denote the deviation of characteristic ratios from their nominal values,

$$\Delta\boldsymbol{\gamma} = \boldsymbol{\gamma} - \boldsymbol{\gamma}^*. \tag{9}$$

$G(s,\ \boldsymbol{\gamma})$ in (7) can be approximated by its first-order Taylor expansion with respect to $\Delta\boldsymbol{\gamma}$.

$$G(s,\ \boldsymbol{\gamma}) = G(s,\ \boldsymbol{\gamma}^* + \Delta\boldsymbol{\gamma}) \approx G(s,\ \boldsymbol{\gamma}^*) + \sum_{i=1}^{n-1} G_i^{'}(s) \frac{\Delta\gamma_i}{\gamma_i^*}. \tag{10}$$

Here $G_i^{'}(s)$ is defined as the sensitivity function of $G(s,\ \boldsymbol{\gamma})$ with respect to γ_i at $\boldsymbol{\gamma}^*$,

$$G_i^{'}(s) = \left. \frac{\partial G(s,\ \boldsymbol{\gamma})}{\partial \gamma_i / \gamma_i} \right|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*}. \tag{11}$$

Combining (7) and (11) gives

$$G_i^{'}(s) = \sum_{j=i+1}^{n} \frac{(j-i) c_j s^j}{(c_n s^n + c_{n-1} s^{n-1} + ... + c_1 s + 1)^2}, \tag{12}$$

for $i = 1,\ 2, ...,\ n-1$, where

$$c_1 = \tau,\ c_k = \frac{\tau^k}{\gamma_{k-1}^* \gamma_{k-2}^{*2} \cdots \gamma_1^{*k-1}},\ k = 2, 3, ..., n. \tag{13}$$

Similarly, the time response of the system in (10) can be expressed as

$$g(t,\ \boldsymbol{\gamma}) = g(t,\ \boldsymbol{\gamma}^* + \Delta\boldsymbol{\gamma}) \approx g(t,\ \boldsymbol{\gamma}^*) + \sum_{i=1}^{n-1} g_i^{'}(t) \frac{\Delta\gamma_i}{\gamma_i^*}, \tag{14}$$

where $g_i^{'}(t)$ is the time response of $G_i^{'}(s)$, namely the sensitivity response of γ_i. As an example, Fig. 1 illustrates the sensitivity responses of a sixth-order system (n=6) taking a unity τ, and the peak amplitudes of the responses, $max|g_i^{'}(t)|$, can be chosen to quantify the influences of γ_i's. Table II summarizes the quantified influences of the characteristic ratios for the third- to eighth-order systems. The quantified influences are normalized such that the sum of the influences is unity in an individual system.

III. Optimization problem

For a high-order system it is challenging to derive the parameters of the low-order controllers directly following the nominal CRA. Even though the high-indexed characteristic ratios have a smaller influence, a significant deviation from

TABLE II
QUANTIFIED INFLUENCES OF CHARACTERISTIC RATIOS

Order	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7
3	77.29%	22.71%	-	-	-	-	-
4	64.17%	28.23%	7.60%	-	-	-	-
5	62.84%	25.11%	10.00%	2.05%	-	-	-
6	63.32%	24.99%	9.14%	2.22%	0.34%	-	-
7	63.40%	25.02%	9.12%	2.10%	0.33%	0.03%	-
8	63.35%	25.06%	9.14%	2.10%	0.32%	0.03%	0.00%

their nominal value (i.e., two) may still largely affect the final time response or lead to instability. Further limitations arise from the existence of zeros and the requirement on robustness. A general design scheme that can simultaneously take care of the requirements on stability, time response, and robustness is necessary, as well as the optimization of all characteristic ratios instead of only optimizing the most important three characteristic ratios, γ_1, γ_2, and γ_3. Thanks to the clear physical meanings of $\boldsymbol{\gamma}$ and τ in time domain, it is straightforward to formulate the optimization problem. This aspect is discussed as follows.

A. Objective function

For a given control plant $\boldsymbol{p}$ and a predefined controller configuration with parameters $\boldsymbol{k}$, the characteristic ratios and the generalized time constant can be expressed as functions of $\boldsymbol{p}$ and $\boldsymbol{k}$ based on the closed-loop transfer function,

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{k},\ \boldsymbol{p}) \text{ and } \tau = \tau(\boldsymbol{k},\ \boldsymbol{p}), \tag{15}$$

respectively. Then the objective function $f(\boldsymbol{k},\ \boldsymbol{p})$ of the optimization problem can be defined as a weighted superposition of the absolute amount of the deviations of the characteristic ratios from their respective nominal values in (8),

$$\begin{aligned} f(\boldsymbol{k},\ \boldsymbol{p}) &= \sum_{i=1}^{n-1} W_i \frac{|\Delta\gamma_i(\boldsymbol{k},\ \boldsymbol{p})|}{\gamma_i^*} \\ &= \sum_{i=1}^{n-1} W_i \frac{|\gamma_i(\boldsymbol{k},\ \boldsymbol{p}) - \gamma_i^*|}{\gamma_i^*}, \end{aligned} \tag{16}$$

where the values of weighting coefficients, W_i's, take the quantified influences of the characteristic ratios in Table II. The objective function defined in (16) emphasizes the damping performance of the closed-loop control system.

B. Constraints

Trade-offs among damping, stability, speed of response, and robustness are essential in controller design. In addition to the requirement on damping indicated by the objective function in (16), other three criteria are considered as constraints in the optimization problem.

1) Stability: It is difficult to directly use Routh-Hurwitz stability criterion when discussing the stability based on the characteristic ratios. Instead, a sufficient condition for the stability, based on the Lipatov-Sokolov stability criterion has been proposed by Manabe and is applied here [4], [10].

$$\gamma_i(\boldsymbol{k},\ \boldsymbol{p}) > 1.12\left(\frac{1}{\gamma_{i+1}(\boldsymbol{k},\ \boldsymbol{p})} + \frac{1}{\gamma_{i-1}(\boldsymbol{k},\ \boldsymbol{p})}\right), \tag{17}$$

TABLE III
BREAK FREQUENCIES (RAD/SEC) OF CHARACTERISTIC POLYNOMIAL UNDER NOMINAL CRA.

Order	ω_{d1}^*	ω_{d2}^*	ω_{d3}^*	ω_{d4}^*	ω_{d5}^*
3	1.3473	2.4506	-	-	-
4	1.4503	3.1494	4.2755	-	-
5	1.4264	3.2855	5.3539	7.8851	-
6	1.4251	3.2436	5.4105	9.9729	15.7748
7	1.4252	3.2428	5.3668	10.0191	20.0308
8	1.4252	3.2429	5.3667	9.9397	20.1312

in which

$$\gamma_n(\boldsymbol{k},\ \boldsymbol{p}) = \gamma_0(\boldsymbol{k},\ \boldsymbol{p}) = \infty, \tag{18}$$

for i=1, . . . , n-1.

2) Speed of response: Pole-zero interaction may limit the feasible choices of the generalized time constant τ for general non-all-pole systems [1], [7]. According to Ref. [1], a more general mechanism of the assignment of τ is given here. For a general closed-loop transfer function,

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_0}{a_n s^n + a_{n-1}s^{n-1} + ... + a_0}, \tag{19}$$

the break frequencies of the denominator polynomial $D(s)$ and nominator polynomial $N(s)$ are the frequencies at which the slopes of the bode plots of the two polynomials are with the unit of 20 dB/decade. The break frequencies can be approximately determined in this way [11]. However, the accuracy of this approximation is problematic due to the complex roots of characteristic equations, i.e., $D(s)$, under the nominal CRA. Due to the generality of the nominal CRA, it is useful to accurately derive the break frequencies of $D(s)$, ω_{di}^*'s, assuming a unity $\tau(= 1sec)$ [1]. ω_{di}^*'s in the third- to eighth-order systems are calculated and listed in Table III. For a non-unity τ, the break frequencies ω_{di}'s are scaled by the value of τ [refer to (4)],

$$\omega_{di} = \frac{\omega_{di}^*}{\tau(\boldsymbol{k},\ \boldsymbol{p})} \text{ for } i = 1, ..., n-1. \tag{20}$$

Similarly, the break frequencies of the nominator polynomial $N(s)$ in (19), ω_{ni}'s, can be obtained either through accurate derivation or approximation. As illustrated in Fig. 2, in order to avoid the resonant peak, i.e., a monotonically decreasing asymptotic magnitude plot of $G(s)$ in Fig. 2(b), and thus to avoid the overshoot in time domain, the following inequalities should be satisfied,

$$\omega_{d1} \leq \omega_{n1},\ \omega_{d2} \leq \omega_{n2},\ \omega_{d3} \leq \omega_{n3}, ... \tag{21}$$

Then from (20) the constraint on the speed of response, i.e., τ, can be defined as

$$\tau(\boldsymbol{k},\ \boldsymbol{p}) \geq max\left\{\frac{\omega_{d1}^*}{\omega_{n1}},\ \frac{\omega_{d2}^*}{\omega_{n2}},\ ...\right\}. \tag{22}$$

Satisfying this constraint will lead to monotonic step responses and thus provides a good starting point of the controller design.

3) Robustness: The uncertainty of the control plant $\boldsymbol{p}$ can be expressed as

$$\boldsymbol{p}_0 - \Delta\boldsymbol{p} \leq \boldsymbol{p} \leq \boldsymbol{p}_0 + \Delta\boldsymbol{p}, \tag{23}$$

where $\boldsymbol{p}_0$ and $\Delta\boldsymbol{p}$ represent the nominal model and the modeling error, respectively. The constraint on robustness can then be defined as

$$\underset{\boldsymbol{p}}{max}\left|\frac{f(\boldsymbol{k},\ \boldsymbol{p}) - f(\boldsymbol{k},\ \boldsymbol{p}_0)}{f(\boldsymbol{k},\ \boldsymbol{p}_0)}\right| \leq \eta, \tag{24}$$

where η is the so-called robust index that quantifies the requirement on robustness, and $f(\boldsymbol{k}, \boldsymbol{p})$ is the objective function in (16). In real applications the value of η should be determined based on the target performance and design requirements of robustness. Obviously the smaller the value of η the higher the requirement on robustness. Moreover, the constraints on stability and speed of response in (17) and (22) must also be guaranteed when parameters vary. Thus the constraints on those two criteria are modified as follows,

$$\underset{\boldsymbol{p}}{max}\left[\frac{1.12}{\gamma_{i-1}(\boldsymbol{k},\ \boldsymbol{p})} + \frac{1.12}{\gamma_{i+1}(\boldsymbol{k},\ \boldsymbol{p})} - \gamma_i(\boldsymbol{k},\ \boldsymbol{p})\right] < 0, \tag{25}$$

$$\underset{\boldsymbol{p}}{max}\left[max\left(\frac{\omega_{d1}^*}{\omega_{n1}},\ \frac{\omega_{d2}^*}{\omega_{n2}},\ ...\right) - \tau(\boldsymbol{k},\ \boldsymbol{p})\right] \leq 0. \tag{26}$$

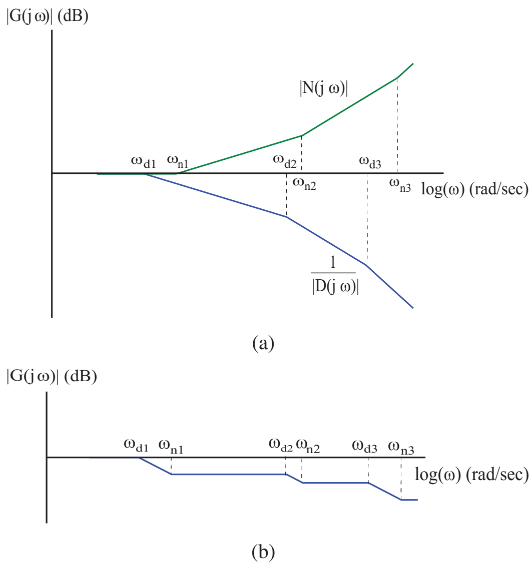

Fig. 2. Asymptotic bode plots for a general non-all-pole system. (a) Nominator and denominator polynomials. (b) Overall transfer function.

C. Problem formulation and solution

Putting the objective function in (16) and the three constraints in (24)–(26) together, the so-called robust optimization problem can be finalized as

$$\begin{aligned}
\underset{\boldsymbol{k}}{min} \quad & f(\boldsymbol{k},\ \boldsymbol{p}_0) = \sum_{i=1}^{n-1} W_i \frac{|\gamma_i(\boldsymbol{k},\ \boldsymbol{p}_0) - \gamma_i^*|}{\gamma_i^*}, \qquad (27)\\
s.t. \quad & \underset{\boldsymbol{p}}{max}\left[\frac{1.12}{\gamma_{i-1}(\boldsymbol{k},\ \boldsymbol{p})} + \frac{1.12}{\gamma_{i+1}(\boldsymbol{k},\ \boldsymbol{p})} - \gamma_i(\boldsymbol{k},\ \boldsymbol{p})\right] < 0,\\
& \underset{\boldsymbol{p}}{max}\left[max\left(\frac{\omega_{d1}^*}{\omega_{n1}},\ \frac{\omega_{d2}^*}{\omega_{n2}},\ ...\right) - \tau(\boldsymbol{k},\ \boldsymbol{p})\right] \leq 0,\\
& \underset{\boldsymbol{p}}{max}\left[\frac{f(\boldsymbol{k},\ \boldsymbol{p}) - f(\boldsymbol{k},\ \boldsymbol{p}_0)}{f(\boldsymbol{k},\ \boldsymbol{p}_0)}\right]^2 - \eta^2 \leq 0,\\
& \boldsymbol{p}_0 - \Delta\boldsymbol{p} \leq \boldsymbol{p} \leq \boldsymbol{p}_0 + \Delta\boldsymbol{p}.
\end{aligned}$$

In (27) the variations of model parameters exist in all three constraints, leading to a nested optimization structure [?]. Each constraint function includes a so-called inner optimization problem with respect to the variations of model parameters $\boldsymbol{p}$. Here all three inner optimization problems are used to guarantee that the constraints on stability, speed of response and robustness should always be satisfied with any possible value of model parameters $\boldsymbol{p}$. Meanwhile, the outer optimization evaluates the performance of each possible solution of $\boldsymbol{k}$ using the objective function. It is a typical robust optimization problem which can be solved via inner-outer robust optimization approaches [?], [12].

In this paper, genetic algorithm (GA) is used as the solver of the outer problem with respect to $\boldsymbol{k}$ (i.e., candidate $\boldsymbol{k}$'s), while the three inner problems are optimized with respect to $\boldsymbol{p}$ by using Matlab fmincon interior-point algorithm, i.e., $\boldsymbol{k}$ is treated as a vector of constants in the inner optimization problems. GA is one of the most popular population-base heuristic approaches that can be used to find global or at least near-to-global optimal solutions. Given the nature of the robust optimization problem in (27), it is appropriate to apply GA to solve the outer problem. In the fmincon function an efficient gradient based optimization method is used to quickly find the optimal solutions, which makes it appropriate for the inner optimization problems. Final candidate $\boldsymbol{k}$'s are the ones that satisfy the three corresponding constraints in (27). The achieved optimal $\boldsymbol{k}^*$ is robust in terms of both stability and time response (speed and shape) with the variations of model parameters. For the present optimization problem, it takes less than 30 seconds to converge using the Matlab tools.

IV. A Case Study

Here a three-mass benchmark system is used as a case study to verify the proposed controller design scheme. This control problem is challenging because the control plant is a fifth-order one and it has two pairs of $j\omega$-axis zeros, two pairs of $j\omega$-axis poles, and one pole at the origin of the s-plane. So far different methods such as disturbance observer (DOB), state feedback, intelligent control, and model predictive control have been proposed for the control of three-mass systems. On

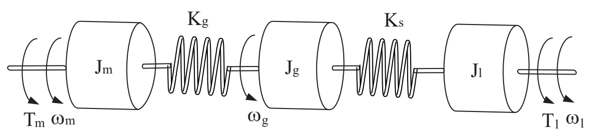

Fig. 3. A simplified three-mass linear model.

TABLE IV
PARAMETERS OF THE NOMINAL THREE-MASS MODEL.

J_{m0}	8.35×10^{-4} $(Kg \cdot m^2)$
J_{g0}	2.40×10^{-3} $(Kg \cdot m^2)$
J_{l0}	3.90×10^{-3} $(Kg \cdot m^2)$
K_{g0}	750 (Nm/rad)
K_{s0}	39.2 (Nm/rad)

the other hand, the PID-based low-order controllers are still predominant in industry. It is both theoretically and practically important to further improve the design of these low-order controllers.

A. Modeling

A simplified three-mass model is shown in Fig. 3. The drive torque T_m is transmitted from the driving side to the load side through two thin torsional shafts. Three inertias are J_m, J_g, and J_l; K_g and K_s are elastic coefficients of the two torsional shafts; T_l denotes the load disturbance torque; ω_m, ω_g, and ω_l are rotary velocities. The linear three-mass model of the torsional test bench can be derived as

$$M(s) = \frac{\Omega_m(s)}{T_m(s)} = \frac{(s^2+\omega_{a1}^2)(s^2+\omega_{a2}^2)}{J_m s(s^2+\omega_{r1}^2)(s^2+\omega_{r2}^2)}. \quad (28)$$

The two resonant frequencies, ω_{r1} and ω_{r2}, and two anti-resonant frequencies, ω_{a1} and ω_{a2}, in (28) are the functions of the model parameters. Here a vector $\boldsymbol{p}$ is defined to denote the parameters of the fifth-order three-mass model,

$$\boldsymbol{p} = [J_m,\ J_g,\ J_l,\ K_g,\ K_s]. \quad (29)$$

and the nominal parameters of the model are listed in Table. IV, namely

$$\boldsymbol{p_0} = [8.35\times10^{-4},\ 2.40\times10^{-3},\ 3.90\times10^{-3},\ 750,\ 39.2]. \quad (30)$$

Thus the nominal anti-resonant and resonant frequencies are

$$\omega_{a1} = 97.50\ rad/s, \qquad \omega_{a2} = 568.26\ rad/s,$$
$$\omega_{r1} = 147.94\ rad/s, \qquad \omega_{r2} = 1099.20\ rad/s, \quad (31)$$

respectively.

B. Controller Design

The so-called m-IPD (modified-Integral-Proportional-Derivative) controller is adopted for the velocity control of the three-mass system, as shown in Fig. 4. The advantage of this special PID controller is that it does not introduce any additional zero into the closed-loop transfer function [refer to

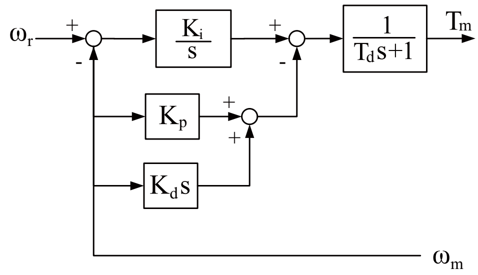

Fig. 4. m-PID control configuration.

(33)].The vector of the control parameters $\boldsymbol{k}$ can be defined as,

$$\boldsymbol{k} = [K_p,\ K_i,\ K_d,\ T_d]. \quad (32)$$

Thus the closed-loop transfer function of a seventh-order system can be derived as

$$G(s) = \frac{\Omega_m(s)}{\Omega_r(s)} = \frac{K_i(s^2+\omega_{a1}^2)(s^2+\omega_{a2}^2)}{\sum_{i=0}^{7} a_i(\boldsymbol{k},\ \boldsymbol{p})s^i}, \quad (33)$$

where the coefficients $a_i(\boldsymbol{k},\ \boldsymbol{p})$ are determined by the control parameters $\boldsymbol{k}$ in (32) and model parameters $\boldsymbol{p}$ in (29),

$$\begin{aligned}
a_0(\boldsymbol{k},\ \boldsymbol{p}) &= K_i\omega_{a1}^2\omega_{a2}^2,\\
a_1(\boldsymbol{k},\ \boldsymbol{p}) &= K_p\omega_{a1}^2\omega_{a2}^2,\\
a_2(\boldsymbol{k},\ \boldsymbol{p}) &= K_d\omega_{a1}^2\omega_{a2}^2 + K_i\omega_{a1}^2 + K_i\omega_{a2}^2 + J_m\omega_{r1}^2\omega_{r2}^2,\\
a_3(\boldsymbol{k},\ \boldsymbol{p}) &= K_p\omega_{a1}^2 + K_p\omega_{a2}^2 + J_mT_d\omega_{r1}^2\omega_{r2}^2,\\
a_4(\boldsymbol{k},\ \boldsymbol{p}) &= K_d\omega_{a1}^2 + K_d\omega_{a2}^2 + J_m\omega_{r1}^2 + J_m\omega_{r2}^2 + K_i,\\
a_5(\boldsymbol{k},\ \boldsymbol{p}) &= J_mT_d\omega_{r1}^2 + J_mT_d\omega_{r2}^2 + K_p,\\
a_6(\boldsymbol{k},\ \boldsymbol{p}) &= J_m + K_d,\\
a_7(\boldsymbol{k},\ \boldsymbol{p}) &= J_mT_d. \quad (34)
\end{aligned}$$

Thus the characteristic ratios and the generalized time constant can be represented as functions of $\boldsymbol{k}$ and $\boldsymbol{p}$,

$$\gamma_i(\boldsymbol{k},\ \boldsymbol{p}) = \frac{a_i^2(\boldsymbol{k},\ \boldsymbol{p})}{a_{i-1}(\boldsymbol{k},\ \boldsymbol{p})a_{i+1}(\boldsymbol{k},\ \boldsymbol{p})}, \text{for i} = 1, ..., 6, \quad (35)$$

$$\tau(\boldsymbol{k},\ \boldsymbol{p}) = \frac{a_1(\boldsymbol{k},\ \boldsymbol{p})}{a_0(\boldsymbol{k},\ \boldsymbol{p})}. \quad (36)$$

Since ω_{a2} is much larger than ω_{a1} in (31), the break frequencies of the numerator of the closed-loop transfer function in (33) are approximately equal to,

$$\omega_{n2} = \omega_{a1},\ \omega_{n4} = \omega_{a2}. \quad (37)$$

Then the constraint on the generalized time constant τ in (27) can be represented as [refer to Table III]

$$\max_{\boldsymbol{p}} \left[max\left(\frac{\omega_{d2}^*}{\omega_{a1}},\ \frac{\omega_{d4}^*}{\omega_{a2}}\right) - \tau(\boldsymbol{k},\ \boldsymbol{p})\right] \leq 0. \quad (38)$$

Here as an example, both the maximum modeling error and the robust index η are taken as 50%, i.e.,

$$\Delta\boldsymbol{p} = 0.5\boldsymbol{p_0} \text{ and } \eta = 0.5. \quad (39)$$

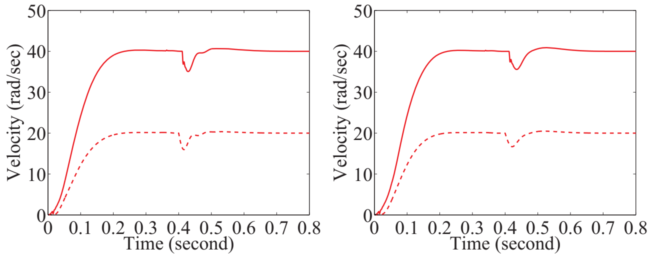

Fig. 5. Velocity responses with variations in model parameter. (a)$\Delta J_l = -0.2420 J_{l0}$. (b)$\Delta J_l = 0.2420 J_{l0}$.

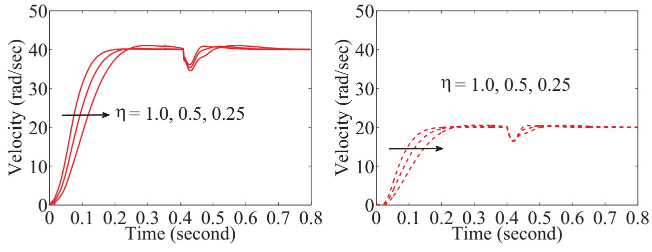

Fig. 6. Velocity responses with different robust index $\eta = 1.0,\ 0.5,\ 0.25$. (a) drive velocity ω_m. (b) load velocity ω_l

Finally, the m-IPD controller is designed through the optimization-based assignment of the characteristic ratios. The results of the optimization are as follows,

$$\begin{aligned} \boldsymbol{k}^* &= [K_p,\ K_i,\ K_d,\ T_d] \\ &= [2.8895,\ 31.4125,\ 0.0966,\ 0.1730], \end{aligned} \quad (40)$$

$$\begin{aligned} \gamma^* = [2.4795,\ 2.5529,\ 2.0952,\ 1.2734,\ 10.1006,\ 0.3636], \\ \tau^* = 0.0920. \end{aligned} \quad (41)$$

It is interesting to note that the value of γ_1, the most important characteristic ratio, is still approximate to its nominal value, while other higher-indexed characteristic ratios are optimized considering the complicated interactions among the limitations of the control configuration, pole-zero interactions, and the robust requirement.

C. Simulation Results

The robustness of the controller is first verified by introducing variations in the model parameter J_l (assuming a 1:2 gear ratio) and a disturbance torque, -5Nm, from 0.4 s. As shown in Fig. 5, good robustness is achieved that validates the optimized CRA obtained through robust optimization. The designed control parameters in (40) provide a good starting point for further improvement of the control performance. The robust index η in (27) is introduced to quantify the requirement on the robustness when solving the optimized CRA. As an example, the results with η =1.0, 0.5, and 0.2, are shown in Fig. 6. The larger the value of η, the more aggressive the controller design and thus the faster the time response, and vice versa [refer to (24)]. It can be seen that the classical tradeoff between the robustness and the speed of response, i.e., the control bandwidth, is well represented and quantified by the robust index η.

V. Conclusions

In this paper, a design scheme for an optimized CRA is proposed for the low-order controller design, especially when the order of the control plant is high. First the influences of the characteristic ratios are quantified as weight coefficients in the objective function of the proposed robust optimization problem, which is defined to minimize the difference between the actual CRA (i.e., damping) and its nominal form. In addition, the requirements on the stability, speed of response, and robustness are also included and represented as constraints. This robust optimization problem is formulated and solved in the inner-outer robust optimization structure. As a case study, the m-IPD controller for the fifth-order three-mass system is designed, and is preliminarily validated by simulation. Compared with existing methods, the proposed scheme explicitly represents the three basic requirements in the controller design, i.e., stability, time response, and robustness. It enables a balanced controller design that is especially superior in robustness again parameter variation and disturbance torque.

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